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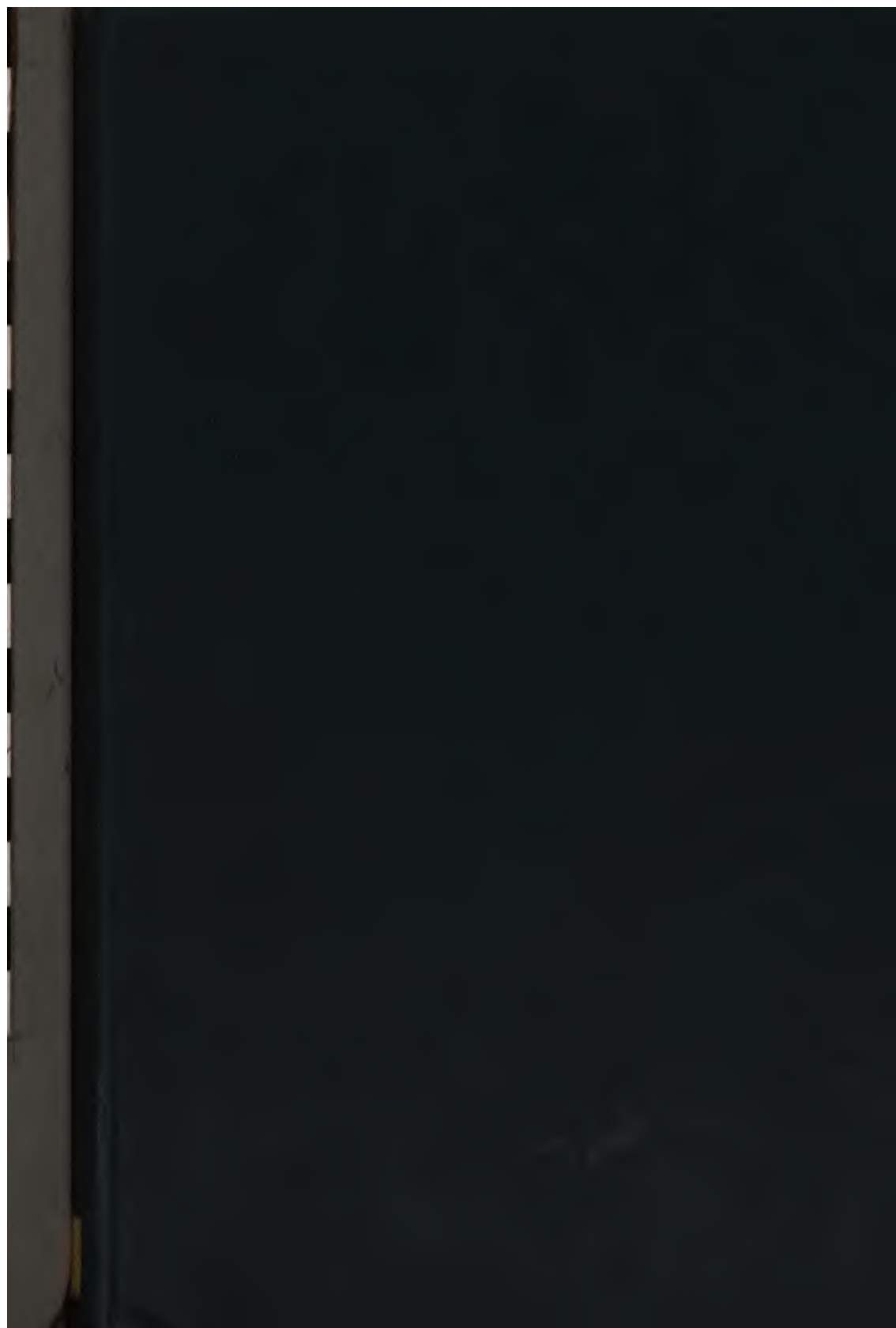
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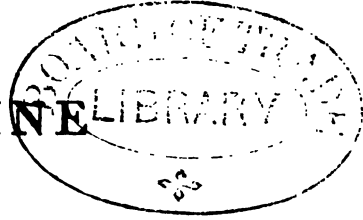
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THE
DOCTRINE
OF
LIFE-ANNUITIES
AND
ASSURANCES,
ANALYTICALLY INVESTIGATED
AND
PRACTICALLY EXPLAINED.
TOGETHER WITH SEVERAL USEFUL
TABLES
CONNECTED WITH THE SUBJECT.
TO WHICH IS NOW ADDED
AN APPENDIX,
CONTAINING
A NEW METHOD
OF CALCULATING AND ARRANGING SUCH TABLES.

IN TWO VOLUMES.

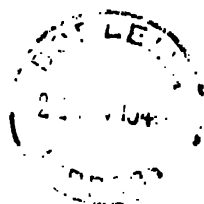
VOL. I.—Containing the *Analysis*.

By FRANCIS BAILY.

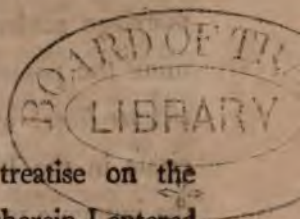
L O N D O N :

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1813.



P R E F A C E.



IN the year 1808, I published a treatise on the *Doctrine of Interest and Annuities*, wherein I entered into a full investigation of all the principles relative to that science; together with its application in the various questions arising from any commercial, political, or financial inquiries. In the preface to that work I signified my intention of prosecuting the subject still further, so as to take in the whole *Doctrine of Life-Annuities and Assurances*: the present treatise, therefore, must be considered as a continuation of the work above alluded to; and will, I believe, contain all that is useful or interesting on the science. The motives, which induced me to submit the *former* work to the public, were there fully explained; and will equally, if not more forcibly, apply to the *present* treatise.

The importance of the subject, at the present day, cannot be doubted; since the greater part of the property of this kingdom is, in one shape or another, connected with this science. The present possessors

of *entailed* estates are, in the common law, justly called *tenants for life*; and the same appellation may be given to those who hold by *curtesy* or by *dower*; marriage settlements also, and wills, generally determine the possession and reversion of estates to *particular lives*: and to these contingencies every freehold estate in the kingdom is liable. If to these we add the immense number of *copyhold* estates determinable on lives, and the estates possessed by *ecclesiastical* persons of every description (all of which will probably be ever subject to the same tenure), we shall find that the value of the greater part of the *real* estates in this country will be determinable upon the principles laid down in the present work.

The incomes, likewise, annexed to all places, civil and military; all pensions, and most charitable donations;—these, and others of a like kind, are annuities for *life*. Moreover, the dividends arising from a great part of the capital in the Public Funds are, by the wills of the donors, and from other causes, rendered of the same nature. Besides which, many *life-annuities* have been granted by Individuals, by Parishes, by Corporate Bodies, and by the Government itself. So that a great part of the *personal* estate, also, of this country is involved in a consideration of this subject.

In addition, however, to the cases above alluded to, there are various other circumstances in which this science will be found highly interesting and useful. There are many parents, at the present day, who are desirous of providing *Endowments* for their children, against they arrive at particular periods of life, when a sum of money is most frequently wanted: such as the time of their apprenticeship, or when they come of age, &c. Several of the Assurance Offices, lately established in London, have published the rates at which they will guarantee such sums: and the present work will enable the public to determine how far it may be *prudent* to accept them. See page 361.

Another interesting part of this subject is connected with the various establishments in this country, under the two general divisions of Societies for the benefit of *Old Age*, and Societies for the benefit of *Widows*. These establishments, when founded and conducted on a true and proper basis, ought always to be encouraged: and can only be objected to when the management of their concerns is likely to fall into the hands of ignorant or designing men, who may be induced to sacrifice the permanent interest of the society to their own immediate benefit and advantage. The ruin of most of these societies may be attributed to

their ignorance or neglect of the true mathematical principles upon which they ought to proceed; and, without an attention to which, no establishment of this kind can possibly flourish.

But, the most important branch of this science is that of *Assurances*: which is still more extensive than either of those above mentioned. For, independent of the different classes of persons holding property under the several tenures alluded to in the beginning of this preface, and whose incomes will consequently determine with their lives, there is an immense number of other persons, in the different departments of society, subject to the same contingency. Every man engaged in either of the three *professions*, whose emoluments arise from his own personal abilities and exertions,—every one pursuing a *naval* or *military* life, whose income will cease at his death,—every person engaged in *manufactures*, *commerce*, or any other employment, whose own immediate exertions are the support of the concern in which he is engaged,—these and many others, too numerous here to insist on, will often be desirous of sacrificing some part of their present emoluments and profits, not only with a view to secure a suitable provision for their families at their decease, but likewise to render their own lives more easy and comfortable,

under the pleasing consolation that they have guarded against one of the great evils of a premature death.

Independent, however, of this general view of the subject, there are various other purposes for which Assurances are effected. Persons holding *Leases on lives*, and paying a *fine* on renewal, are oftentimes induced to insure a sum of money upon those lives, in order that they may be enabled to pay such fine when it becomes due. Some consider it a good method of securing a dubious or protracted *debt*, by assuring the life of the debtor. Others, again, may be entitled to an estate, or to a sum of money, at the end of a given term, or on the happening of a particular event, provided they be then alive to receive it: and, in order to secure such sum to their families, may be desirous of insuring their lives for such *term*, or against such *contingency*.

These, and a thousand other cases of daily occurrence, render this branch of the science interesting to every class of the community. Numerous Offices have lately sprung up, in the metropolis, for the purpose of granting Assurances on every possible contingency amongst lives in general; and it therefore becomes every one, engaged in the public business of life, to study this subject with attention.

But, notwithstanding the importance and utility of these inquiries, it is not much more than a century that they have been conducted in a proper and scientific manner*. The celebrated Dr. Halley led the way in England; and in his paper, inserted in the *Philosophical Transactions* for 1693, pointed out the true method of calculating the value of Annuities on *Lives*. In the pursuit of this object, he assumed the rate of human mortality for five successive years, as observed at *Breslaw*; and, from these data, formed the *first* correct table of the value of Life-Annuities. That table, however, being adapted only to every *fifth* year of human life, and calculated at only *one* rate of interest, was consequently very limited in its application and utility.

The illustrious De Moivre improved on what Dr. Halley had begun†. He carefully examined the table

* Soon after the Revolution in this country many of the *loans*, for the service of Government, were raised upon Life-Annuities: and nothing can show more forcibly the low state of the science, at that period, than the vague manner in which the values of such annuities were estimated.

† Abraham De Moivre was born at Vitri, in Champagne, in 1667. The revocation of the Edict of Nantz, in 1685, determined him, with many others, to take shelter in England; where he perfected his mathematical studies, the foundation of which he had laid in his own country, and which have rendered him so great an ornament to the age in which he lived. In the latter part of his life he subsisted chiefly by giving answers to questions in Chances,

of observations given by that celebrated philosopher: and, finding that for several years together the decrements of life were uniform, and that it was only in youth and in old age that any considerable deviation occurred, he founded his ingenious hypothesis that the decrements of life are *equal* and uniform, from birth to the utmost extremity of human life. He was, at first, inclined to compose a Table of the values of Life-Annuities, by keeping close to the table of observations; that is, by dividing the whole extent of human life into *several* intervals, according to the difference of the decrements during those periods. But, before he undertook this task he tried what would be the result of supposing those decrements *uniform* from the age of twelve to the utmost extremity of life: and was satisfied that the excesses, arising on one side, would

Annuities, &c: and it is said that most of these solutions were delivered at a coffee-house in St. Martin's-Lane, where he spent the greatest part of his time. His merit and abilities were so well known and esteemed, that the Royal Society of London judged him a fit person to decide the famous contest between **NEWTON** and **LEIBNITZ**, concerning the invention of Fluxions. He was highly esteemed by the first of these celebrated philosophers: and it is reported that, during the last ten or twelve years of **NEWTON**'s life, when any person came to ask him for an explanation of any part of his works, he used to say "Go to M. De Moivre; he knows all these things better than I do!" He died at the advanced age of eighty-seven.

be compensated by the defects on the other. For, on comparing his calculation with that of Dr. Halley, he found the conclusions to differ so very little, that he thought it superfluous to join together several different rules in order to compose a single one.

The first edition of his *Annuities on Lives* was printed in octavo, in 1724. By the most simple and elegant formulæ he pointed out the method of solving all the most common questions relative to the value of Annuities on single and joint lives, Reversions and Survivorships. In the subsequent editions of that work* he not only corrected the errors into which he

* The *second* edition appeared in 1743 ; and the *third* in 1750. Since which time, I believe there have been other editions; but the most improved copy is that which is inserted at the end of his *Doctrine of Chances*, 3rd edit. 1756.

In the preface to the second edition, here alluded to, he made an illiberal and unjustifiable attack on Mr. Simpson ; and charged him with mutilating his propositions, obscuring his demonstrations, and pirating his rules. But Mr. Simpson *effectually* refuted these charges (in the same year) in an *Appendix* to his *Doctrine of Annuities* : at the close of which he exclaims, in the language of conscious rectitude, " I appeal to all mankind, whether, in his treatment of me, he has not discovered an air of self-sufficiency, ill-nature, and inveteracy, unbecoming a gentleman." Here the controversy appears to have dropped. For, M. De Moivre published the third edition of his book, without any further notice of Mr. Simpson : but omitted the offensive reflections which had been inserted in the preface to the preceding edition.

had fallen in the first edition, but also greatly enlarged the boundaries of the science, and encouraged other mathematicians to pursue the path which he had struck out with so much honour to himself. Unfortunately, however, his hypothesis will not suit all circumstances: and more recent discoveries, on the rate of human mortality, have proved that it cannot always be safely adopted. Nevertheless it is still of great use in the investigation of many cases connected with this subject; and will ever remain a proof of his superior genius and abilities.*

In the year 1742, Mr. Thomas Simpson published the first edition of his little treatise on the *Doctrine of Annuities and Reversions*; in which he introduced the method of computing such values from the *real observations* of life. His rules upon this subject are general, and will apply to any observations: nevertheless he confined himself, in his Tables and in his Examples, to the rate of human mortality in *London* as deduced from the observations of Mr. Smart†. The same author prosecuted this subject, by way of *Sup-*

* The Tables of the value of Life-Annuities, deduced from M. De Moivre's hypothesis, will be found at the end of the present work; being Tables XLIX and L.

† Mr. Simpson's Tables of the value of Life-Annuities will be found at the end of this work; being Tables XLV to XLVIII.

plement, in his *Select Exercises for young Proficients in the Mathematics*, published in the year 1752.* This work, however, is, for the most part, a repetition of the rules given in the preceding treatise: to which are added some new problems on the subject of *contingent annuities and assurances*.

On the style of Simpson (always simple and elegant) it is needless for me to make any observations. His works are his best comment; and, need only be read, to be admired. Nevertheless his *Treatise on Life-Annuities*, together with his *Supplement* above alluded to, are perhaps the most imperfect of his productions. Although much is there done, still much more remained to be executed. His *Tables*, being deduced from the rate of mortality in *London* only, are found not to be sufficiently adapted for general use: and his *Rules*, being deduced partly from the hypothesis of M. De Moivre and partly from real observations, have been ascertained not to be sufficiently correct. Subsequent improvements in the science have also shown that some of his general theorems are

* That part relating to Annuities has lately been taken out of the *Select Exercises*; and, having been printed separately in 1791, is now generally bound up with his *Doctrine of Annuities and Reversions*: the second edition of which appeared in 1775, and which contains the *Appendix* alluded to in the note in page xii.

erroneous: and that many cases, which frequently occur in practice, are not even mentioned in either of his works.

In 1753, Mr. James Dodson published the Second volume of his *Mathematical Repository*; in which are contained not only the algebraical solutions of the problems given by M. De Moivre in his treatise above mentioned, but also several new and useful Questions connected with this subject: the Third volume of the same work appeared two years afterwards (1755). In the compass of two small duodecimo volumes, the author has contrived to solve an immense variety of Questions relative to Annuities, Reversions, Survivorships, and Assurances. The methods which he has pursued in investigating these cases are, in general, a model of analytical reasoning; and afford an excellent *praxis* for the young mathematician. Nevertheless he is sometimes obscure, from the use of uncouth symbols; and from the culpable practice of changing the signification of his characters during the course of the same investigation. In all his solutions he adopted the hypothesis of his friend De Moivre; conceiving that it would lead to more accurate results than the use of the Table of Life-Annuities formed by Mr.

Simpson (from the bills of mortality in *London*), the only one, at that time, deduced from real observations. The reader therefore will look in vain for any *correct* solution in the works of this author: although they may be occasionally referred to, for the method of finding an *approximate* value.

The science remained in this state, without much improvement, till the publication of the first edition of Dr. Price's celebrated treatise in 1769. This work, entitled *Observations on Reversionary Payments, &c.*, was first published with a view to oppose and destroy the injurious effects and evil intentions of a class of men (unfortunately to be found in every stage of society) who, under pretence of establishing Societies for the *benefit* of Old Age, and of Widows, were only forming schemes to allure, and to defeat the hopes of the ignorant and the distressed. His efforts were eventually crowned with success: and those *bubble* societies have long since met with the fate which he so truly predicted.

In this laudable pursuit, Dr. Price saw the necessity of more accurate observations on the mortality of human life; in order to determine with more correctness the value of Life-Annuities, and to show more forcibly

the futility and extravagance of the schemes that were issued by those societies*. By the assistance of some public-spirited individuals he obtained correct registers of the rate of mortality at Northampton, Norwich, Chester, and other places in England. But, still, the computation of the values of annuities, according to these observations, was a work so tedious and unpleasant, that little hopes were entertained of profiting by those researches: and Dr. Price suffered three several editions of his treatise to pass over without affording any additional information on this subject. At length the fourth edition appeared (1783) enriched with several valuable tables of Annuities on Single and Joint lives, at different rates of interest, deduced not only from the probabilities of living as observed at *Northampton*, but also from the probabilities of living as observed in the kingdom of *Sweden* at large†.

* Any person, who will take the trouble to go through the Examples inserted in Dr. Price's treatise, will readily observe how inaccurately he was obliged to proceed in this infant state of the science. In calculating the value of *deferred* annuities, (a case of frequent occurrence,) he was obliged to take the *value of the annuity* from M. De Moivre's tables, but, the *probabilities of life* he deduced from Dr. Halley's table of observations at Breslaw. A practice which gives an air of imperfection to the work at the present day; and which *ought* to have been removed, after the publication of the late valuable tables.

† These tables are inserted at the end of the present work;

The great addition which Dr. Price has made to our means of information respecting this science, and the assiduity with which he thus promoted some of the best interests of mankind, deserve the highest commendation: and his labours on this subject entitle him to our warmest praise. The primary object, which he had in view, has been fully answered; and his treatise was admirably adapted to *that* end. In every other respect, however, it is far from being complete: and the reader will look in vain for the most common cases that occur in practice. Indeed, those subjects, which are to be met with, do not readily present themselves; owing to the loose and irregular manner in which they are treated. Dr. Price's object was not so much to insert what was *new*, as to illustrate (by some striking examples) a few of the leading problems, with a view to oppose the pernicious schemes that disgraced the age in which he lived. But, those schemes having long since vanished, his observations may now be considered rather as a *beacon* to posterity*.

being Tables XIV to XLIV. Their publication forms a new *era* in the science; and has, in a great measure, rendered obsolete and useless the preceding works of De Moivre, Dodson, Simpson and Morgan. This latter treatise had been published about four years before this period; as will be mentioned in the sequel.

* Any person, the least acquainted with the subject of the pre-

The next treatise on this subject is that by Mr. Morgan, entitled *The Doctrine of Annuities and Assurances*, which appeared in 1779. This author sets out with the vain attempt to render the principles of the science intelligible to persons *unacquainted* with mathematics: but, after a fruitless effort for this purpose, he ultimately leaves his readers to pursue their inquiries by the common and only useful method of *analysis*. Besides some valuable observations “on the different methods of determining the state of a Society, whose business consists in making Assurances on Lives,” that work will be found to contain a variety of problems treated, for the most part, in a plain, easy and familiar manner; and adapted to the state of the science at *that* period. But, out of the forty-two problems which that treatise contains, about thirty of them, chiefly relating to *contingent* annuities and assurances, are (owing to more accurate observa-

tion, must be aware that any additional *Tables* of the value of Life-Annuities, or any *Observations on the best method of forming them*, will add greatly to our means of information. It will therefore readily be seen that my remarks do not allude to this part of his treatise, which I consider invaluable, and of constant utility. My observations, in the present instance, apply more particularly to any improvement in the *analysis* of the science; and its application to any practical cases.

tions and a more improved analysis) now rendered totally unfit for general use. Mr. Morgan himself, however, has been the principal cause of this revolution in the science: but of the *merit* of his improvements on this subject I shall speak hereafter*.

The last professed treatise on the science, which I think worthy of notice†, is Mr. Baron Maseres's *Prin-*

* In Mr. Morgan's *Doctrine of Annuities &c*, we find three new tables of the value of Life-Annuities, deduced from the probabilities of life as observed at Northampton: namely, one for Single lives, another for two Joint lives whose ages are equal, and another for two Joint lives whose difference of age is sixty years: the interest in each table being at 4 per cent. In this infant state of the science, every additional table contributed greatly to the means of information on this subject.

It may be here necessary to remark that the fourth edition of Dr. Price's *Obs. on Rev. Pay.* (which first contained the present valuable collection of Tables) did not appear till four years after the publication of Mr. Morgan's work above alluded to. So that, till within these thirty years, there existed only four tables of the value of Life-Annuities: viz. two founded on M. De Moivre's hypothesis, and two deduced from the *London* observations; which are the Tables XLVII to L at the end of this work. As to the table formed by Dr. Halley, it was too contracted for any real use.

† I must not however omit mentioning the name of Dr. Waring: who, in a small pamphlet entitled *On the Principles of translating Algebraic Quantities into Probable Relations, &c* (1792), has devoted about thirty pages to the consideration of the subject of Annuities and Assurances. His style and manner, however, will not be much admired by those who have read the works of preceding writers on this subject. The formula in page 61, is taken from that pamphlet.

Principles of the Doctrine of Life Annuities (1783): wherein this celebrated author has explained the subject in so familiar a manner, as to be intelligible even to those who are unacquainted with the Doctrine of Chances, and who have made no great proficiency in mathematics. This treatise, however, (although consisting of more than 700 quarto pages) goes no further, in the analysis of the subject, than the first two problems in the present work: but its value is greatly enhanced by containing a variety of new Tables of the value of Annuities on Single lives, and on two Joint lives of different ages, deduced from the probabilities of living as observed by *M. De Parcieux* amongst the Government Annuitants in France; these being justly considered by the learned author as the most proper data whereon to found the value of Life-Annuities*. There are, moreover, in that treatise several interesting observations on the best method of providing Annuities for *Old Age*, and on various subjects of finance and political economy; which render it particularly valuable to those who are desirous of information on these

* These tables (with the exception of those on Single lives, at the several rates of 2, $2\frac{1}{2}$, 7, 8, 9, and 10 per cent interest) are inserted at the end of the present work; being Tables V to XIII. The original tables, however, are carried to six places of decimals:—a dubious effort!

important questions, and will perpetuate the name and abilities of this truly public-spirited writer.

Soon after the publication of the *fourth* edition of Dr. Price's *Observations on Reversionary Payments*, (which contained the valuable collection of Tables of Life-Annuities, deduced from the observations made at Northampton and in Sweden) Mr. Morgan was enabled to detect the inaccuracy of those rules which, not only Mr. Simpson and others had given for determining the value of *contingent* annuities and assurances, but also which he himself had deduced from the same principles, in his treatise above mentioned: and he immediately set about to correct them. His labours, on this subject, are contained in the several *papers* inserted by him in the *Philosophical Transactions* for 1788, 1789, 1791, 1794 and 1800. In the first volume, here alluded to, he has considered those cases only in which *two* lives are concerned: in the next two volumes, his object was to deduce the value of contingent assurances in all those cases where *three* lives are concerned, and which admit of a *correct* answer; and in the last two volumes he proposed to determine the value of contingent annuities and assurances in all the *remaining* cases of *three* lives.

Whoever will take the pains to read over those

papers with attention, must be struck with surprise and regret at the *strange* and *confused* manner which Mr. Morgan has pursued, in order to obtain the solution of the several problems under consideration. No one, at the present advanced state of the science (with so many models of simplicity and elegance before him), could expect to see any mathematical inquiries conducted in so loose, so obscure, and so extraordinary a manner. The investigations are tediously and unnecessarily prolix; crowded with useless repetitions, and a variety of unmeaning quantities. All which might, indeed, be excused, if the resulting *formulae* had been at once simple and correct: instead of which, we find the grossest errors committed, not only as to their *form*, but as to their *accuracy*. They are, for the most part, *unnecessarily* long; abounding with *useless* quantities, (which render their numerical solution exceedingly intricate and difficult); and oftentimes *at variance* with the particulars mentioned in the investigation: which, together with the *erroneous* manner in which they are printed, renders them of little or no use to the public. Most of his problems are investigated in *two* different ways, and are solved by the means of *two* distinct *formulae*: but, notwithstanding the similarity of these methods is studiously

kept from the observation of the reader, and although these double formulæ are, in each problem, totally different *in appearance*; yet they will be found in *all* cases to be precisely the same *disguised under different symbols*! A curious and interesting branch of the science has been thus strangely distorted and enveloped in mystery; a depraved taste in mathematical reasoning has been introduced; and (what is, by far, of the greatest importance) many *false solutions* have probably resulted from too great a dependance on the general formulæ*.

Mr. Morgan and myself are the only persons that have ever yet attempted to give correct solutions in the several cases of *Contingent Annuities and Assurances*. These cases have been fully investigated in the fifth and eighth Chapters of the following treatise. But, in conducting those investigations, I could not avoid a frequent reference to the preceding labours of Mr. Morgan on this subject; not only with a view of

* The *Philosophical Transactions* not being within the reach of every person, Mr. Morgan has inserted his *formulæ*, for the solution of the several problems here alluded to, in the last edition of Dr. Price's *Obs. on Rev. Pay.* note (P). But, the errors of the *original* are multiplied in the *copy*: and Mr. Morgan, if he studies his own reputation as a mathematician, had better expunge them altogether in future, than suffer them to remain in their present *worse than useless* state.

censuring the culpable method which he has adopted in pursuing his inquiries; but also, in order to obviate any objection that might be made to *my* formulæ, *because* they do not correspond with *his*. It is needless, however, in this place, for me to add to the comments which I have already made in the two chapters above alluded to*.

The above are the principal English† authors that

* See the *Observations* in pages 195, 214, and 241; also the several *Notes* in pages 186, 187, 210, 257, 259, 261, 263, 273, 279, 288 and 308: where the charges, above insisted on, are fully explained and demonstrated. See likewise the *Notes* in pages 114, 118, 126 and 137.

† With respect to the *foreign* writers on this science, their productions are more numerous than ours, but their inquiries are not so extensive. The subject of Life-Annuities was treated by Van Hudden of Amsterdam; and likewise by the celebrated Jean de Witt, in his treatise entitled *De vadye van de lijfrenten &c* (1671). M. Struyck also inserted, in the Introduction to his *Universal Geography* (1740), some conjectures on the state of human mortality; and a long treatise on the method of calculating the value of Life-annuities. But M. Kerseboom carried his researches much further, in his treatise published in 1748; and afterwards in 1752.

Whilst these inquiries were pursuing in Holland, M. De Parcieux was occupied with the same subject in France. In his *Essai sur la Probabilité de la Durée de la Vie Humaine* (1746) he has endeavoured to establish the rate of mortality which exists amongst Life-Annuitants *only*; and has adopted it as a proper standard for determining the value of Life-Annuities. But besides this important point, he has discussed a number of other interesting subjects connected with this science: and his work will be read with much profit and advantage. In 1779 M. St. Cyran published his

have written on the subject of Life-Annuities and Assurances. They are few in number; and the whole of their productions, taken collectively, by no means contain a complete view of the science. And, more-

Calcul des Rentes Viageres, which contains many useful and valuable Tables. M. De Parcieux (the nephew of the preceding author of the same name) published also a treatise on this subject, entitled *Traité des Annuités* (1781). But the most useful work on this science is that published by M. Duvillard, under the title of *Recherches sur les Rentes &c* (1787).

The researches of M. Wargentin and M. Sussmilch are well known in this country, from the frequent mention of their labours by Dr. Price, in his *Obs. on Rev. Pay.* The immortal EULER has also condescended to illustrate the first principles of this science in a paper inserted by him in the *Histoire de l'Acad. Roy. de Berlin* for 1760: wherein a method is given (similar to that of Mr. Simpson) for determining the value of an annuity on a life one year younger, from the value of an annuity on a life one year older. The same author has likewise inserted, in his *Opuscula Analytica* (1785), the solution of a question relative to Reversionary Annuities.

But, notwithstanding the list of authors which is here adduced, it will be found that, as far as the *analysis* of the subject is concerned, the science remained nearly stationary under their hands. Their inquiries, in this respect, were confined principally to the method of deducing the value of annuities, on single and joint lives, from given tables of observations; that is, to such subjects as are detailed in the second Chapter of the present work. Those useful and interesting parts of the science which relate to the subject of Reversions, Survivorships, and Assurances, together with their several applications to the various purposes of life, do not enter into any of the foreign treatises which I have had an opportunity of seeing.

over, the late improvements have rendered them, in a great measure, either obsolete or useless: and have shown the necessity of a general revision of the subject. Under these circumstances I was induced to form a new treatise, which should comprehend not only all that is useful and important in either of the preceding works, but also such additional information as a more improved analysis and more recent discoveries in the science have been able to afford.

The following is the outline of my plan.

The First chapter contains a few elementary principles of the *Laws of Chance*; some remarks on the *Probabilities of Life*, with an account of the several *Tables of Observations* made at different parts of the world; and an explanation of the general method adopted to express those probabilities in all cases. This preliminary chapter will prevent much unnecessary repetition in the course of the work.

The Second chapter shows the method of determining the *Value of Annuities* on any *Single* or *Joint* lives; on the *Longest* of any number of lives; &c, &c. The second corollary to the first problem is of considerable importance in enabling us to deduce, in a very easy and expeditious manner, the value of annuities, on any single or joint lives, from real obser-

vations*. For, it should be particularly observed that, Tables of such values being once formed, the solutions to the subsequent problems become extremely easy; since the formulæ are expressed in terms denoting the value of such annuities.

The Third chapter contains the four necessary problems for the solution of all cases of absolute *Reversionary annuities*: and at the end of that chapter I have selected all the possible cases of two and three lives, in order that they may be more easily referred to. The formulæ there given will be found of considerable utility also in enabling us to determine the value of the *Fines* that ought to be paid for the *Renewal of Leases* held on two or three lives: as I have fully explained in the Examples given in page 397.

The Fourth chapter comprehends various cases of annuities depending on *Survivorships* between two and three lives. These cases might have been considerably augmented, but without any real benefit: since the most frequent ones are there inserted; and any other (which may arise) is easily solved by the same method of proceeding.

* The *Specimen*, inserted in page 36, is not the same as that given by Dr. Price in his *Obs. on Rev. Pay.*, but an improved method deduced from the formula there alluded to.

The Fifth chapter relates to such cases of *Contingent Reversionary Annuities* as could not, for want of some previous information, be inserted in the two preceding chapters: and I believe that the method of solution, which I have there adopted, will come nearer to the correct value than any that has hitherto been published.

The Sixth chapter treats of *Assurances*: a subject of great importance and extensive utility at the present day. A full explanation of the doctrine is given in the two problems inserted in that chapter.

The Seventh chapter contains the method of determining the value of annuities on *Successive lives*; the value of *Fines* in Copyhold estates held on lives; the value of *Presentations*, *Advowsons*, and things of a like kind. It likewise enables us to determine the value of the *Fines* that ought to be paid for Renewing or Exchanging any lives held on a lease originally granted for three lives and afterwards for a number of years certain: a practice pursued by several corporations in this country*.

The Eighth chapter is devoted to an investigation of

* See some singular errors and absurdities, into which the Corporation of *Liverpool* had fallen upon this subject, pointed out in page 418.

the value of *Contingent Assurances*; wherein I have considered every possible case in which not more than three lives are concerned. In this branch of the science I flatter myself that I have made considerable improvements. I have divested the subject of all extraneous matter; have not introduced more cases than were absolutely necessary; have exposed the singular formulæ given by Mr. Morgan (the only person who has preceded me in these inquiries); and have, for the most part, introduced more correct expressions for the value of the several cases there alluded to.

The three remaining chapters complete the *analysis* of the science, and relate to such subjects as could not properly be introduced into either of the preceding ones. The Ninth is confined to an explanation of the celebrated *hypothesis* of M. De Moivre: wherein its great utility and convenience, in many obvious cases, is defended against the recent attacks of Dr. Price and Mr. Morgan. The Tenth treats of the method of determining the value of life-annuities payable *Half-yearly, Quarterly, &c.*: also of the value of life-annuities *Secured by Land*: and of the value of Assurances of sums of money payable *immediately* on the extinction of any given lives. The Eleventh shows the method of finding in *Annual Payments* the value of

any Assurance or of any Deferred Annuity: problems which will be found of very extensive use in practice.

The Twelfth chapter contains a variety of very useful *Questions* connected with this subject: to which are added the *Rules* for the solution of the same; and a numerous collection of *Examples*. These are thrown together into one chapter for two obvious reasons: in the first place, by being separated from the body of the work, they do not interrupt the analytical investigations; and secondly, they may be used (together with the *Tables* which follow) by such persons as are not acquainted with mathematics. Consequently, the present work will be accommodated to the use of both classes of readers; and (although some repetition is unavoidably occasioned thereby) may be thus rendered doubly valuable. The questions in this chapter are such as most frequently occur: but, others of less public utility, or the solution to which could not be conveniently expressed in words at length, are to be met with in the body of the work, subjoined to the respective problems.

The Thirteenth chapter shows the direct application of the sixth, thirteenth and eighteenth questions, in the preceding chapter, to some of the most useful and

important concerns of life: namely, to the method of forming the best *Schemes* for providing annuities for the benefit of *Old Age*, and for *Widows*. These observations are brought together under one head, in order that they might not interrupt the regular arrangement of the questions: and because it gives me, thereby, an opportunity of enlarging more fully on this very interesting subject.

The Last chapter contains an account of the various *Societies* or *Companies* that have been formed in this metropolis for the purpose of granting Annuities, Assurances and Endowments; together with a review of their several plans and constitutions. By this statement the world may be enabled to judge of their comparative merits, and will easily decide on their respective claims to public patronage. This additional chapter cannot be considered foreign to the nature of the present work. For, after explaining the principles of the science, and after entering at large into the value of such subjects, it becomes me not only to show *where* a person might apply who is desirous of negotiating for such things, but likewise to point out those Offices at which he may do it with the greatest *security* and *advantage* to himself. And I am the more induced to do this, as I have lately observed

several mean attempts to mislead and deceive the public. Uninterested in, and unconnected with, any of these Societies, I have been anxious only to give a clear and unprejudiced account, deduced from their own plans and proposals, as submitted by them to the public*. At the end of this chapter I have also inserted a brief account of the recent plan, adopted by Government, for converting the 3 per cent stocks into Life-Annuities. The propriety and advantage of a measure of this kind, I had suggested in my *Doctrine of Interest and Annuities*.

Respecting the *Tables* of Observations, and of Life-

* Whilst the present work was at the press, I struck off 500 copies of this Fourteenth Chapter, which I published as a pamphlet, in a separate form, under the title of *An Account of the several Life-Assurance Companies established in London*. The whole impression has been already sold: but I may probably reprint it at some future opportunity.

From the tenor of that pamphlet many persons have imagined that I have given an unqualified approbation of the plan and constitution of the *Equitable Society*: whereas my intention was merely to represent it as the *most* equitable and advantageous society *now existing*. The exclusion (except in very rare instances) of persons, assured *for a term of years*, from the benefits of the Society, as well as the use of disproportionate rates for a *particular class* of the assured, are not consonant to the original views and intentions of the Society. Moreover, the long interval between each division of profits, as well as the *mode* of dividing those profits, are not such as (in my opinion) ought to be adopted by any *new society*: although we might justly pause before any alteration could be safely proposed for the adoption of the *Equitable Society* in its present state.

Annuities deduced therefrom, which are inserted at the end of this work, I can only observe that they are a collection of all that have hitherto been published, in this country, on the subject: and that they are here carefully reprinted from the authorities mentioned in the note in page 41. The necessity of *other* tables I have often had occasion to mention; but I fear it will be long before I shall be able to congratulate the public on this point. It consequently becomes the more necessary to have all those, which *do* exist, brought into one view, in order that they may be more conveniently consulted. The Tables of the value of Assurances for Single and Joint Lives, according to the observations made by *M. De Parcieux* and in *Sweden*, have been calculated expressly for this work; and will be oftentimes found extremely useful.

It now remains only to explain and to justify the method of notation adopted throughout the present work: and (since it is, for the most part, novel in its nature) to obviate any objections that may be made to its introduction into the analytical science. It is needless for me, at the present day, to point out the great advantage attending a convenient and simple notation: since it is notorious to every one, conversant with mathematics, that some of the greatest improvements in the science (and in the most abstruse parts of it, too)

have arisen from this very source. The expressions for the probabilities of life, inserted in § 23, are an improvement on the method adopted by Mr. Simpson. For, he denotes the *probabilities* of the life A continuing 1, 2, 3, &c, years, by the characters $\overset{1}{a}$, $\overset{2}{a}$, $\overset{3}{a}$, &c: whereas I make $\overset{1}{a}$, $\overset{2}{a}$, $\overset{3}{a}$, &c, to denote the number of *persons living* at the end of those years, whereby the probabilities of the life A continuing 1, 2, 3, &c years, will be represented by the *fractions* $\frac{\overset{1}{a}}{a}$, $\frac{\overset{2}{a}}{a}$, $\frac{\overset{3}{a}}{a}$, &c. This method of denoting those quantities keeps the several parts distinct, which will be found of considerable use in many of the investigations which arise in the progress of the work. In the next place, I have selected the Greek characters α , β , γ , to denote the number of persons living *at an age n years older than* A, B, C, respectively; whereby the solutions of all cases relative to Deferred or Temporary annuities are rendered extremely simple, and easily to be remembered: all the investigations of such cases, by preceding authors, being rendered extremely confused and unscientific, for want of some general principle of this kind*. I cannot anticipate any objection to this measure.

* See particularly the methods pursued by Dr. Price and Mr. Morgan in *Obs. on Rev. Pay.* notes (B), (C) and (D).

Throughout the whole of this work I have uniformly represented the lives, which are the subject of discussion, by the *Roman* capitals A, B, C, &c; reserving the *Italic* capitals *A*, *B*, *C*, &c, to denote the *value of an annuity* on those lives. This improvement, though apparently trifling, is however more correct than the uniform practice, hitherto pursued, of making the same letters denote two different quantities in the same investigation. I have also adopted a more accurate mode of expression, when speaking of the value of annuities on lives, than has hitherto been followed by preceding authors. I never use the ambiguous phrase *the value of a life*, when I mean to express the *value of an annuity* on such life: neither do I, when speaking of more than one life, leave it doubtful whether the expression alludes to the *joint* lives, or to the *longest* of such lives. These vague and equivocal expressions ought to be banished from all true science; and, they would scarcely have been considered worthy of refutation, had they not been so generally adopted by *every* author who has hitherto written on this subject (not excepting even the accurate Simpson himself): whereby a vicious and corrupt mode of expression has been introduced, without any corresponding advantage to compensate for gross an error.

Since every series, which expresses the value of an annuity on the whole continuance of life, may be divided into two parts, which (separately taken) denote the value of a Deferred and Temporary annuity, and since those cases are of frequent occurrence, it becomes exceedingly desirable to be enabled to express such partial series by some general character. The method which I have adopted has been to enclose, in a parenthesis, the value of an annuity on the given lives; and to affix the Roman letter d or t (according as the annuity is Deferred or Temporary), in order to show that the series, represented by the quantity within such parenthesis, is not complete. This plan I have found, in a variety of cases, extremely useful and convenient. See pages 43 and 45.

In deducing the value of Deferred annuities, however, in any practical questions, it is always requisite to know the value of similar annuities on the same number of lives *older by the given term* than such lives: it is therefore useful to be enabled also to denote the value of such annuities by some convenient expression; as well in order to avoid circumlocution as to render the resulting formulæ more simple and elegant. Such values I have, throughout the following

work, denoted by affixing the small Italic letter *o* to the characters denoting the value of annuities on the given lives. Whereby these new values may be easily distinguished from the values of annuities on the original lives; and their meaning may be easily retained in the memory. See page 43.

With respect to the new characters introduced in the twenty-seventh and twenty-ninth Problems, (see pages 186, 189, 190, and 211) they are adopted more for the sake of a convenient reference than for any analytical purpose. With this view, I have, in the use of them, chosen such as will most readily assist the memory: the life on which the assurance is made being always expressed by the largest letter; the smaller ones being added merely to denote the number and value of the lives on which the contingency depends. Thus, the value of an assurance on the life *A*, provided he dies the first of *two* lives *A B*, is denoted by A^B : but, if it depends on his dying the first of *three* lives *A B C*, it is denoted by A^{BC} . In like manner, the value of an assurance on the life *B*, provided he dies the first of *two* lives *B C*, is denoted by B^C : but, if it depends on his dying the first of *three* lives *A B C*, it is denoted by B^{AC} . For a

similar reason also I have, in the thirty-fifth Problem, (see page 240) denoted the value of an assurance on the decease of A or B, provided *either* of them be the first that dies of three lives A B C, by the character AB^C . It will be found that many of the problems in the eighth chapter may be expressed in a very simple manner by means of these symbols; which can be my only apology for introducing them. Mr. Morgan (who is the only person, besides myself, that has attempted to give correct solutions to such problems) has arbitrarily adopted, in *his* formulæ, the unmeaning letters E, Σ , M, Q, &c, to denote the values here alluded to. But, when compound quantities are represented by more simple expressions, those characters ought to be preferred which will, most readily and with the least effort of memory, bring to our recollection the original quantity intended to be expressed*:

* This principle has guided me in the choice of the characters inserted in the Scholium to the first Lemma in the fifth chapter (see page 116). I mention these things merely to obviate the objections of some hyper-critics, who may probably complain of the work as being *disfigured by Greek characters, and uncouth symbols*.

Si forte necesse est
Indiciis monstrare recentibus abdita rerum,
Fingere cinctatis non exaudita Cethegis
Continget; dabiturque licentia sumta pudenter.

Hor.

and I should have been much better satisfied if I could have pursued the same plan in the thirty-ninth and subsequent Problems.

Such is the nature of the present work, which will most probably terminate my labours on this subject. Much of my time is taken up in answering questions, which are laid before me for solution, relative to Annuities and Assurances. Those solutions are oftentimes different from such as arise from the ordinary rules and methods laid down by preceding writers: and it is on this account that I have been more particular in my inquiries on this subject; as well as desirous of explaining the cause of the difference, in order to remove any doubt as to their accuracy or propriety. The theorems, from which my practical rules are deduced, are strictly and mathematically demonstrated in the course of the present work: and in the *numerical* enunciation of those rules (when applied to the solution of such cases as are submitted to my consideration) I discard the *indiscriminate* use of the Life-Annuity Tables, deduced from the *Northampton* Observations; so generally adopted by the different Assurance Offices, and so much recommended by their immediate supporters.

The motives, which have influenced me to this determination, it is unnecessary here to enter into: since they are fully explained in the course of the present work. And I can only add that they will continue to be my rule of conduct, as long as I am appealed to, by the public, as an arbiter on these subjects.

FRANCIS BAILY.

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Feb. 12, 1810.*

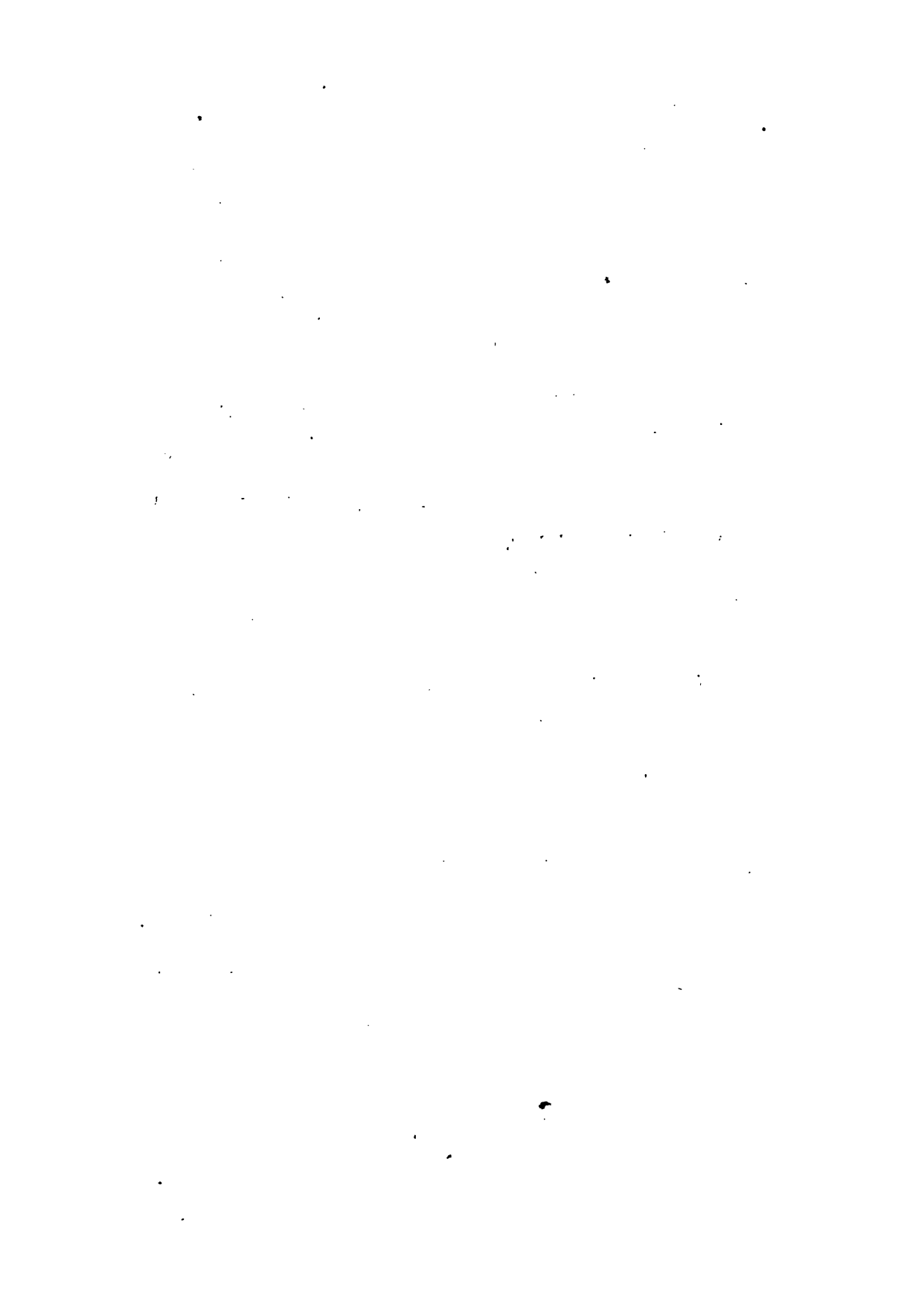


TABLE OF CONTENTS.

CHAP.	Page
1. <i>On the Laws of Chance, and Probability of Life</i>	1
2. <i>On Life Annuities in general</i>	25
3. <i>On Reversions</i>	63
4. <i>On Survivorships</i>	74
5. <i>On Contingent Annuities</i>	106
6. <i>On Assurances</i>	142
7. <i>On Successive Life-Annuities, &c.</i>	163
8. <i>On Contingent Assurances</i>	181
9. <i>On M. De Moivre's Hypothesis</i>	313
10. <i>On Half-yearly &c Annuities; &c.</i>	331
11. <i>On Annuities and Assurances in Annual Payments</i>	348
12. <i>Practical Questions</i>	355
13. <i>On Schemes for the benefit of Old Age and Widows</i>	462
14. <i>On the several Life-Assurance Companies</i>	479
TABLES	521

THE
DOCTRINE
OF
LIFE ANNUITIES,
&c.



CHAPTER I.

*On the Laws of Chance; and the Probability of
Human Life.*

§ 1. IT is not my intention here to enter into a full investigation of the nature and laws of chance, but merely to explain those principles of the doctrine which are more essentially connected with the subject of the present work; in order to prevent any misunderstanding in the terms which are occasionally made use of.

§ 2. The *probability* of the happening of any event is to be understood as the ratio of the chances, by which that event may happen, to all the chances by which it may either happen or fail: and it may be expressed by a fraction whose numerator is the number of chances whereby the event may happen, and whose denominator is the number of chances whereby it may

either happen or fail. Thus, if there be a chances for the happening of any event, and b chances for its not happening; then will the probability of such event taking place be truly represented by $\frac{a}{a+b}$.

§ 3. In like manner, the probability of any event failing (or of its not happening) may be expressed by a fraction whose numerator is the number of chances whereby it may fail, and whose denominator is, as before, the whole number of chances whereby it may either happen or fail. Thus, the probability of the above event failing will be truly expressed by $\frac{b}{a+b}$.

§ 4. Since the sum of the two fractions, representing the probabilities of the happening and of the failing of any event, is equal to unity; it follows that, one of them being given, the other may be found by subtraction. Thus, the probability of an event happening being denoted by $\frac{a}{a+b}$, the probability of the same event failing will be truly represented by $1 - \frac{a}{a+b} = \frac{b}{a+b}$; and *vice versa*.

§ 5. If, upon the happening of an event, a person be entitled to a given sum of money, his *expectation* of receiving that sum has a determinate value before the happening of the event; and such value is ascertained by multiplying the present value of the sum

expected by the fraction which represents the probability of obtaining it. Thus, if a person has a chances of obtaining, and b chances of losing a certain sum of money, the present value of which is equal to s , then will $s \times \frac{a}{a+b}$ denote his expectation of receiving such sum, and will be the true value of his interest therein*.

* These principles may be more familiarly explained by the following example. Suppose that a person has 3 chances in 5 to obtain £100, the present value of his *expectation* is the product of £100 by the fraction $\frac{3}{5}$, and consequently it is worth £60. For, supposing that an event may equally happen to any one of 5 different persons, and that the person to whom it does happen should, in consequence of it, obtain the sum of £100, it is plain that the right which each of them in particular has upon the sum expected is $\frac{1}{5}$ of £100; which right is founded on this principle, that if the 5 persons concerned in the happening of the event should agree not to stand the chance of it, but to divide the sum expected among themselves, then each of them must have $\frac{1}{5}$ of £100 for his pretension. Now, whether they agree to divide that sum equally among themselves, or rather choose to stand the chance of the event, no one has thereby any advantage or disadvantage, since they are all upon an equal footing: and consequently each person's *expectation* is worth $\frac{1}{5}$ of £100. Let us further suppose that two of the five persons concerned in the happening of the event, should be willing to resign their chance to one of the other three; then the person to whom these two chances are thus resigned has now 3 chances that favour him, and consequently he has now a right triple of what he had before, and therefore his expectation will in such case be worth $\frac{3}{5}$ of £100. Now, if we consider that the fraction $\frac{3}{5}$ expresses the probability of obtaining the sum of £100, and that $\frac{3}{5}$ of 100 is the same as $\frac{3}{5} \times 100$, we must naturally fall into the conclusion laid down.

§ 6. The probability of the happening of *several events*, that are independent of each other, is equal to the product of the probabilities of the happening of each event considered separately. Thus, if the probability of the happening of the first of any number of independent events be denoted by $\frac{a}{a+b}$; that of the second, by $\frac{c}{c+d}$; that of the third, by $\frac{e}{e+f}$; &c, &c: then will $\frac{a}{a+b} \times \frac{c}{c+d} \times \frac{e}{e+f} \times \&c$, denote the probability of the happening of all those events. And this expression, multiplied by the present value of the given sum, will denote the value of the *expectation* of receiving such sum on the happening of all those events*.

in the text, that the expectation of receiving any sum is determined by multiplying such sum by the probability of obtaining it: and though this method of reasoning is deduced from a particular case, it will easily be perceived that it is general, and applicable to any other case. See De Moivre's *Doctrine of Chances*, p. 3.

* For example: Suppose that, in order to obtain £100, two events must happen, the first whereof has 3 chances to happen and 2 to fail, and the second whereof has 4 chances to happen and 6 to fail: the value of the expectation will in such case be $\frac{3}{5} \times \frac{4}{10} \times 100 = 24$ pounds. The demonstration of which will be very easy if it be considered that, supposing the first event had happened, the expectation (then depending entirely upon the second) would, before the determination of the second, be worth $\frac{4}{10} \times 100 = 40$ pounds. We may therefore look upon the happening of the first as a condition of obtaining an expectation worth £40: but, the probability of the first event happening has been supposed $\frac{3}{5}$, wherefore the expectation sought for is to be estimated

§ 7. By a similar method of reasoning it will be evident that the probability of the failing of any number of independent events is equal to the product of the probability of the failing of each event considered separately. Thus, if the probability of the failing of the first of any number of independent events be denoted by $\frac{b}{a+b}$; that of the second, by $\frac{d}{c+d}$; that of the third, by $\frac{f}{e+f}$; &c, &c: then will $\frac{b}{a+b} \times \frac{d}{c+d} \times \frac{f}{e+f} \times \&c$, denote the probability of the failing of all those events. And this expression, multiplied by the present value of the given sum, will denote the value of the *expectation* of obtaining such sum on the failing of all those events.

§ 8. Moreover, the probability of the happening of *either* of any number of independent events is denoted by the difference between unity and the expression mentioned in the last article. For, since $\frac{b}{a+b} \times \frac{d}{c+d} \times \frac{f}{e+f} \times \&c$, denotes the probability that any given number of events shall fail, it follows (from § 4) that $1 - \frac{b}{a+b} \times \frac{d}{c+d} \times \frac{f}{e+f} \times \&c$, will denote the probability that they shall *not* all fail; but that

by $\frac{1}{2} \times \frac{1}{10} = 100$; that is, by the product of the two probabilities of happening, multiplied by the sum expected. The same method of reasoning may be applied to the happening of three, or any other number of events, as may be seen more at large in the authors who have treated on this subject.

some one or other of them will happen. And this expression, multiplied by the present value of the given sum, will denote the value of the *expectation* of receiving such sum, on the happening of either of those events.

§ 9. In like manner, if the expectation of receiving any sum depends upon the happening of any number of independent events, and upon the failing of any number of other independent events, its value will be equal to the present value of such sum, multiplied by the probability of all the former happening, and also by the probability of all the latter failing. And from these principles, we may determine the value of an expectation depending on the happening or failing of as many independent events as may be assigned.

§ 10. Hitherto I have considered only such events as are independent of each other: but, if we wish to determine the probability of the happening of two events that are dependent on each other*, we must multiply the probability of the happening of one of

* Two events are independent, when they have no connection with each other, and the happening of one neither forwards nor obstructs the happening of the other: as the continuance or failure of any given lives. On the other hand, two events may be considered as dependent, when the probability of either's happening is altered by the happening of the other; as the continuance or failure of the same life in different periods of its duration: see § 2.

them, by the probability which the other will have of happening when the first is considered as having happened: and the same rule will extend to the happening of as many events as may be assigned.

§ 11. If there are several expectations upon several sums, it is evident that the expectation upon the whole will be equal to the sum of the expectations upon each. But if only one sum is to be received on the happening or failing of the given events, the method of determining the value of the expectation will be somewhat altered. The process, however, which is to be pursued in such cases will be more fully explained in the course of the present work: what has been already said being merely introductory to the various probabilities and contingencies that occur in the following sheets.

§ 12. Now with respect to the probability that a person of a given age will or will not live to any other given age, or till a certain sum of money granted him becomes due, it is obviously in all cases a matter of very great uncertainty, and will be often very different in different persons of the same age. The chance which a man of thirty years of age, who is in good health and lives a temperate and quiet life in the country, has to live twenty years, or till he is fifty years of age, is evidently much greater than that of another man of the same age, and of the same degree of health and vigour of body, who lives in a

great city and in scenes of riot and dissipation; and it is likewise greater than that of another man of the same age and of the same degree of health and vigour, but who is going into an unhealthy climate to which he has not been accustomed: and still more evidently it is greater than that of another man of the same age who is of a weak and sickly constitution, or who by his daily occupation is exposed to many dangers of his life from which the generality of mankind is exempt; as is the case with soldiers and sailors in time of war or actual service. These are circumstances beyond the reach of calculation; and all that can be done by any general rules upon this subject is to estimate the degree of probability with which it may be reasonably expected that a person of any given age will live to any other given age, upon a supposition that he has neither a better nor a worse chance of so doing than the majority of other persons of the same age. This medium or average chance of living is determined by tables that exhibit the number of persons which, out of a certain number of children born (usually not less than a thousand), are found by a long series of observations to be living at the end of every subsequent year of human life to its extreme period: which period in some of the tables is carried to 86, and in others to more than 90 years. The instances of the prolongation of human life to 100 years, or more, are so few that they are not thought to be worth attending to in forming any general rules on this subject.

§ 13. Various observations on the mortality of human life have been made by different persons, and in different places; and several tables of the kind above-mentioned have been calculated and formed by the different writers on this subject, such as Dr. Halley, Mr. Thomas Simpson, M. Kersseboom, M. De Parcieux, Dr. Price, M. Susmilch, M. Wargentin, M. Muret, and others. But the same table of the probabilities of life will not suit every place: for, long experience has shown that all places are not equally healthy; or that the number of persons who die annually is different in different places. Dr. Halley formed his table from observations on the births and burials of the inhabitants of the city of *Breslaw* (the capital of the duchy of *Silesia* in *Germany*), during a series of five years; viz. from 1687 to 1691. Mr. Thomas Simpson, from observations on the bills of mortality in *London* for ten years, from 1728 to 1737*. M. Kersseboom, from the registers of certain assignable annuities for lives in *Holland*, which had been kept there for one hundred and twenty-five years, and in which the ages of the several people dying in that period had been truly entered. M. De Parcieux, from a similar use of the lists of the tontines in *France*,

* In Table I, at the end of the present work, the column denoting the probabilities of life in *London*, is not the one here mentioned; but a more correct one formed by Dr. Price, from the bills of mortality for ten years, from 1759 to 1768. The same author likewise formed one from the bills of mortality for ten years, from 1771 to 1780.

the numbers of which were verified by the *Nécrologes*, or mortuary registers, of several religious houses of both sexes*. Dr. Price, from a register of mortality kept at *Northampton*, for forty-six years, from 1735 to 1780: the same author has also formed a table from a similar register kept at *Norwich* for thirty years, from 1740 to 1769: another from a similar register kept by Mr. Gorsuch at *Holy Cross* near Shrewbury, for thirty years from 1751 to 1780: another from a similar register kept by Dr. Aikin at *Warrington* in Lancashire, for nine years, from 1773 to 1781: another from a similar list kept by Dr. Haygarth at *Chester*, for ten years, from 1772 to 1781: another from the register of mortality at *Vienna*, for eight years: another from the register of mortality at *Berlin*, for four years, from 1752 to 1755: another from a similar register at *Brandenburgh*, for fifty years, from 1710 to 1759: each of the last three being from tables given by M. Susmilch: also another from the tables of mortality at *Stockholm*, for nine years, from 1755 to 1763, as given by M. War,

* In the table alluded to in the last note, the column denoting the probabilities of life in *France*, is not the one given by M. De Moivre and Mr. Ferguson; but a more correct one published by M. Florencourt at the end of his Treatise on Political Arithmetick. Referring however to the last note, it may be necessary here to remark that in Tables III and XLV I have preserved the original values as given by M. De Parcieux and Mr. Simpson; because, the Life Annuity Tables being deduced therefrom, I consider them as the most proper to be used in conjunction therewith.

gentin : and another from seven different enumerations of the whole population of the kingdom of *Sweden*, each repeated at the end of three years, viz. in 1757, 1760, 1763, 1766, 1769, 1772, and 1775. M. Muret formed his table from registers kept in forty-three parishes in the district of *Vaud* in Switzerland, for ten years, from 1756 to 1765.

§ 14. All these tables differ from each other ; and in many cases so materially as to leave us in great doubt whether the subject has attained that degree of accuracy and correctness to which it is capable of being carried. It should be observed that there are two sorts of data for forming tables of the probability of the duration of human life : one is furnished by the registers, or *bills of mortality*, which show the numbers dying at all ages ; the other, by the proportions of deaths at all ages to the numbers living at those ages, as discovered by surveys or *enumerations*. Those tables which are deduced from the former of these data are correct only when there is no considerable fluctuation among the inhabitants of a place, and when the births and burials are equal : for, when there are more removals *from*, than *to* a place, and the births exceed the burials (as is almost always the case in country parishes and villages) tables so formed give the probabilities of living too low : and when the contrary happens (as is generally the case in cities and large towns) they give the probabilities of living too high. But tables formed from the latter of these

data are subject to no errors: they must be correct whatever the fluctuations are in a place, and how great soever the inequalities may be between the births and the burials.

§ 15. Most of the tables above-mentioned have been deduced from the former of these data; and in most of them due allowances have been made, as far as circumstances would admit, for the fluctuations arising from emigration &c. But I believe there are no observations extant which will enable us to form tables from the latter of these data, except those published by M. Wargentin* of the population of the kingdom of Sweden: and it is much to be regretted that similar observations are not made in other countries.

§ 16. It is a singular circumstance that not only do *females* live longer than *males*, but *married women* live longer than *single women*. All the tables of observations intimate this; but the fact has been more fully confirmed by the observations made by Dr. Aikin at *Warrington*, and by Dr. Haygarth at *Chester*; each of whom kept distinct registers of the rate of mortality amongst males and females. Similar registers also were kept at *Stockholm*: and, in the enumeration of the whole population of the kingdom of *Sweden*, this circumstance was particularly attended to. These

* In the *Memoirs of the Academy of Sciences at Stockholm*, in 1776.

latter observations therefore being formed on such unerring principles, furnish sufficient data for calculating distinct tables of the value of annuities on lives among males and females, taken separately or conjunctly: and which tables might be applied with good effect in determining the value of annuities or assurances where the lives of *widows* are concerned*.

§ 17. The tables of observations most used in this country at present, are those which were formed by Dr. Price from the bills of mortality at *Northampton*: but they derive their importance principally from those numerous tables of the value of annuities on single and joint lives, which are computed therefrom; and which afford great facility to the solution of the various cases connected with this subject†. In every other point of view it must appear extremely incorrect to take the rate of mortality in one particular town, as a criterion for that of the whole country. The observations ought to be made on the kingdom at large, in the same manner as in Sweden; more particularly as, in the real business of life, the calculations are general and uniform, and adapted to persons in every situation. But till the legislature thinks proper to adopt some efficient plan for furnishing these data‡,

* See Tables XIV to XXIV.

† See Tables XXV to XLIV.

‡ By virtue of an Act of Parliament which passed in 1800, a survey was made of the population of this kingdom in the following year: but, from the manner in which that survey was formed and conducted, it has afforded little or no information respecting the subject of the present work.

we must rest contented with the laudable exertions of public-spirited individuals; and avail ourselves of the best light which they afford us on this subject.

§ 18. With respect to the several tables of mortality above-mentioned, I do not think that any of them (with the exception of those given by M. Kersseboom, and M. De Parcieux) afford the proper grounds for calculating the value of *annuities*. For, it is evident that no person, in an ill state of health, or who is conscious of any thing in his constitution that might tend to the shortening of life, would give that value for an annuity which the tables indicate: neither am I inclined to think that he would become a purchaser at all. The lives, therefore, of such persons as *do* become annuitants will consequently be *good* lives: or a certain part only of the general mass of mankind. The principles, upon which M. Kersseboom and M. De Parcieux have formed their tables, enable us to ascertain pretty accurately the rate of mortality among this class of people: and therefore form a proper basis for determining the value of *annuities*.

§ 19. The same observations may be applied, with nearly the same propriety, to the method of computing the value of *assurances**. For, it is well known that

* This fact is indubitable: for “ during the last 33 years, from “ January 1768 to January 1801, the number of assurances on “ single lives [at the Equitable Society] has been 83,201; of “ which number sixty thousand five hundred and ninety-seven “ have been on the lives of persons under 50 years of age, among

the assurer endeavours to guard as much as possible against a *bad* life: and the law of the land justly punishes any fraud in this respect. Nevertheless, as avarice or negligence may induce a relaxation of duty, the lives on which assurances are made are more liable to be *mixed* than those on which annuities are granted. In either case, however, we might often deduce a more correct value from knowing the situation of life, the residence, and mode of living of the parties concerned.

§ 20. But in many instances, both of annuities and assurances, the ages and conditions of the lives are so involved that we must proceed upon general

“whom the deaths have been *fewer* than those in the Northampton table in the proportion of four to seven!” (See Dr. Price’s *Obs. on Rev. Pay.* vol. ii. p. 443.) No fact can more clearly show the inaccuracy of those tables for *general* use? and though it may be *prudent* for an *Insurance company* to adopt them, as well as to make use of the lowest rate of interest in calculating the values of annuities therefrom (whereby large profits are secured to the society); yet the *public*, who have no interest therein, and who occasionally seek for information on this head, should be cautious in using them, unless they appear to be applicable to the case in question. The grounds, on which the calculations are made, ought to be as correct as the present state of information will allow, in order that the public may be satisfied with the accuracy of the result. The contrary, however is the fact: and Dr. Price himself has at length acknowledged it; although in rather a surreptitious manner. He introduces M. Kersseboom’s and M. De Parcieux’s tables of observations in order “that nothing on this subject may be wanting;” as if it were a work of supererogation, and not one of the most essential as well as one of the most valuable parts of his treatise.

principles, without reference to the particular situation of the parties: and therefore, were it on this ground only, it would be extremely desirable to ascertain the rate of mortality in the kingdom at large. It would enable us to determine how far the tables, now in use, might be depended upon; and furnish the basis for others more numerous and comprehensive.

§ 21. For the information of the reader I have inserted a comparative view of all the principal tables that have been given of the rate of mortality in different parts of the world; being Table I. at the end of this work. The first column shows the ages, and the other columns the number of persons living at those ages, out of 1000 born* at the different places mentioned at the head of each column: and these places are arranged according to their degree of mortality amongst them. London and other cities are therefore placed first; and the rest in their order, as

* The original tables commence with numbers differing from each other; but are here reduced to the same number at the beginning, viz. 1000: by which mean we are enabled, by inspection, to compare the numbers together at any age, and immediately perceive the relative degrees of mortality at the several places given. The reader will observe that I have given *other* tables of the probabilities of life for *France, Sweden, Northampton* and *London*, together with the decrements or number of persons dying annually; which, being on a more enlarged scale, may be used with greater accuracy in the solution of the several problems which occur in the present work. These Tables are III, XIV, XXV, and XLV.

nearly as possible, to the most healthy, which are the country provinces. This table will consequently serve to illustrate, in a striking manner, the great difference between the duration of life in large *cities* and in the *country*: for it will be seen that, in proportion as we recede from the former, the probability of life is greater, and the chance of arriving at old age is considerably increased. Thus, it appears that, out of a thousand persons born at Vienna, not half of them live to be two years of age; whereas at Norwich, that number will live to be eight years of age; and at Holy Cross, they live to be above twenty-seven years of age; whilst in the province of Vaud in Switzerland, they live to be forty-one years old. It will also fully confirm the observation, which has been made in § 18, respecting the probability of living amongst those persons who purchase annuities on their own lives: for it appears from the observations of M. De Parcieux that the chance of living amongst a set of *government annuitants* is in almost every period of their existence much *greater* than amongst an equal number of indifferent persons *living in the most healthy part of the globe*: and which consequently shows that the Northampton tables are a very inaccurate index of the rate of mortality amongst a set of persons who purchase annuities on their own lives.

§ 22. But, however inaccurate these tables of observations may be, or however inapplicable to existing circumstances, the subject of the present work is not

at all affected thereby. For, since the principles here laid down, and the rules thence deduced, are all treated *generally*, without allusion to any particular table of observations, the reader may apply them to any of the tables above mentioned; or to any others which may be hereafter found to be more correct, or more suited to any given circumstances.

§ 23. In any table of observations, therefore, which expresses the number of persons living at every age of human life, let the number of the living at the age of A be denoted by a ; and those answering to the next succeeding ages in the table, by a' , a'' , a''' , &c, respectively. Further; let the number of the living at the age of B be denoted by b ; and those answering to the next succeeding ages in the table, by b' , b'' , b''' , &c, respectively. Again; let the number of the living at the age of C be denoted by c ; and those answering to the next succeeding ages in the table, by c' , c'' , c''' , &c, respectively. In like manner, let the number of persons living at the end of n years from the age of A be denoted by α ; and those answering to the next succeeding ages in the table, by α' , α'' , α''' , &c. Also let the number of persons living at the end of n years from the age of B be denoted by β ; and those answering to the next succeeding ages in the table by β' , β'' , β''' , &c. Moreover, let the number of persons living at the end of n years from the age of C be

denoted by γ ; and those answering to the next succeeding ages in the table by γ' , γ'' , γ''' , &c. And so on for any other lives. Then, considering that, out of a persons alive at the age of A , only a' of them will be alive at the end of the year, it is evident that the number of chances for the life of A continuing one year will be a' ; and that the whole number of chances for its living or dying will be a : consequently the probability that A will live to the end of the first year will be denoted by $\frac{a'}{a}$. And by a similar method of reasoning it will be seen that the probability of his living to the end of the second year will be denoted by $\frac{a''}{a}$; and of his living to the end of the third year, by $\frac{a'''}{a}$; and so on: for, a'' , a''' , &c, will be respectively the number of chances for the life continuing 2, 3, &c, years, and a will still be the total number of chances for his living or dying in any year. In like manner, the probability of the life B continuing 1, 2, 3, &c, years, will be respectively represented by $\frac{b'}{b}$, $\frac{b''}{b}$, $\frac{b'''}{b}$, &c: and that of the life C , by $\frac{c'}{c}$, $\frac{c''}{c}$, $\frac{c'''}{c}$, &c, respectively: (See § 2). By a similar method of reasoning it will be found that the probability of the life A continuing n , $(n+1)$, $(n+2)$, $(n+3)$, &c, years will be respectively denoted by $\frac{a}{a}$, $\frac{a'}{a}$, $\frac{a''}{a}$,

$\frac{\alpha'''}{a}$, &c: and that the probability of the life B continuing the same periods will be respectively denoted by $\frac{\beta}{b}$, $\frac{\beta'}{b}$, $\frac{\beta''}{b}$, $\frac{\beta'''}{b}$, &c: and that the probability of the life C continuing the same periods will be respectively denoted by $\frac{\gamma}{c}$, $\frac{\gamma'}{c}$, $\frac{\gamma''}{c}$, $\frac{\gamma'''}{c}$, &c.

§ 24. Moreover, the probability of any two lives, A and B, continuing in being together for 1, 2, 3, &c, years, will be respectively denoted by $\frac{a'b}{ab}$, $\frac{a''b}{ab}$, $\frac{a'''b}{ab}$, &c: and that of three lives A, B, C, by $\frac{a'b'c}{abc}$, $\frac{a''b''c}{abc}$, $\frac{a'''b'''c}{abc}$, respectively: (See § 6). Further, the probability of any two joint lives A and B continuing in being together for n , $(n+1)$, $(n+2)$, $(n+3)$, &c, years, will be respectively denoted by $\frac{a\beta}{ab}$, $\frac{a'\beta'}{ab}$, $\frac{a''\beta''}{ab}$, $\frac{a'''\beta'''}{ab}$, &c: and that of three lives A, B, C, for the same terms, by $\frac{a\beta\gamma}{abc}$, $\frac{a'\beta'\gamma'}{abc}$, $\frac{a''\beta''\gamma''}{abc}$, $\frac{a'''\beta'''\gamma'''}{abc}$, &c.*

* I would here observe that, when several lives are involved in a question, A is generally supposed (unless otherwise expressed) to be the *youngest*; B older than A; C older than B; and so on. This rule, however, is dispensed with in the fifth and eighth chapters of the present work.

§ 25. This being premised, it is evident (from § 4) that the probability of A *dying* before the end of the first year will be denoted by $1 - \frac{a'}{a}$. For, since $\frac{a'}{a}$ denotes the probability of his living to the end of that period, if we subtract this value from unity it will give the probability of his not living so long. And by a similar method of reasoning it will be found that $(1 - \frac{a''}{a})$, $(1 - \frac{a'''}{a})$, &c, will denote the probability of the same life dying before the end of the second, third, &c, year, respectively. In like manner the probability of the life B or C failing in 1, 2, 3, &c, years, will be respectively represented by $(1 - \frac{b'}{b})$, $(1 - \frac{b''}{b})$, $(1 - \frac{b'''}{b})$, &c: or $(1 - \frac{c'}{c})$, $(1 - \frac{c''}{c})$, $(1 - \frac{c'''}{c})$, &c. Moreover, the probability that *either* of two lives A, B, or of three lives A, B, C, will fail in 1, 2, 3, &c, years, will be denoted by $(1 - \frac{a'b'}{ab})$, $(1 - \frac{a''b''}{ab})$, $(1 - \frac{a'''b'''}{ab})$, &c; or $(1 - \frac{a'b'c'}{abc})$, $(1 - \frac{a''b''c''}{abc})$, $(1 - \frac{a'''b'''c'''}{abc})$, &c, respectively. And universally, if we subtract from unity the probability of the lives continuing together to the end of the given term, the remainder will express the probability that they shall not

all continue together to the end of that period ; but that one or other of them will die previous thereto.

§ 26. But, the probability that *all* the lives A, B, C, &c, shall fail in one year will (by § 6) be denoted by $(1 - \frac{a'}{a}) \times (1 - \frac{b'}{b}) \times (1 - \frac{c'}{c}) \times \&c$; in two years, by $(1 - \frac{a''}{a}) \times (1 - \frac{b''}{b}) \times (1 - \frac{c''}{c}) \times \&c$; in three years, by $(1 - \frac{a'''}{a}) \times (1 - \frac{b'''}{b}) \times (1 - \frac{c'''}{c}) \times \&c, \&c$. And the probability that this event shall not happen, but that some *one* or other of the lives shall continue in being to the end of the first, second, third, &c, year, will (by § 8) be represented by $1 - (1 - \frac{a'}{a}) \times (1 - \frac{b'}{b}) \times (1 - \frac{c'}{c}) \times \&c$, $1 - (1 - \frac{a''}{a}) \times (1 - \frac{b''}{b}) \times (1 - \frac{c''}{c}) \times \&c$, $1 - (1 - \frac{a'''}{a}) \times (1 - \frac{b'''}{b}) \times (1 - \frac{c'''}{c}) \times \&c$, &c, respectively.

§ 27. Hitherto, in deducing the probability of a life failing in any given time, I have had regard only to such event taking place *at any time before* the end of that period : but if we wish to determine the probability of the life failing *in* any particular year the exegesis will be materially different. The probability that A will die in the second year, after having out-

lived the first year, is evidently equal to $1 - \frac{a''}{a'} = \frac{a' - a''}{a'}$; because $\frac{a''}{a'}$ will *then* denote the probability of its living to the end of that year, and this value, being subtracted from unity, will give the probability of its *then* dying in that year: but since this event *now* depends upon its living through the preceding year (the probability of which is $\frac{a'}{a}$) the value above found must be multiplied by such probability, in order to give its true value: whence, the *present* value of the probability that the life A will fail *in* the second year, is truly denoted by $\frac{a' - a''}{a'} \times \frac{a'}{a} = \frac{a' - a''}{a}$. In like manner the probability of its failing *in* the third year is expressed by $\frac{a'' - a'''}{a''} \times \frac{a''}{a} = \frac{a'' - a'''}{a}$: and so on to the n^{th} ($n + 1$)st, ($n + 2$)nd, ($n + 3$)rd, &c, years; when the probabilities of the given life A failing *in* those years will be respectively $\frac{a - a'}{a}$, $\frac{a' - a''}{a}$, $\frac{a'' - a'''}{a}$, $\frac{a''' - a''''}{a}$, &c: and where a denotes the number of persons living at the end of $n - 1$ years from the age of A. The same observations will apply to the case of any number of joint lives: for, by pursuing the same method of reasoning, it will be found that the present probability of *either* of the three lives A, B, C, failing *in* the second year will

be denoted by $\frac{a'b'c' - a''b''c''}{a'b'c'} \times \frac{a'b'c'}{abc} = \frac{a'b'c' - a''b''c''}{abc}$; in

the third year, by $\frac{a''b''c'' - a'''b'''c'''}{a''b''c''} \times \frac{a''b''c''}{abc} = \frac{a''b''c'' - a'''b'''c'''}{abc}$;

and so on to the n^{th} , $(n+1)^{\text{st}}$, $(n+2)^{\text{nd}}$, $(n+3)^{\text{rd}}$, &c, year; when the probabilities of the lives failing

in those years respectively will be $\frac{a\beta\gamma - a'\beta'\gamma'}{a\beta\gamma}$, $\frac{a'\beta'\gamma' - a''\beta''\gamma''}{a'\beta'\gamma'}$,

$\frac{a''\beta''\gamma'' - a'''\beta'''\gamma'''}{a''\beta''\gamma''}$, $\frac{a'''\beta'''\gamma''' - a''''\beta''''\gamma''''}{a'''\beta'''\gamma'''}$, &c: and where $a\beta\gamma$ de-

note the number of persons living at the end of $n-1$ years from the ages of A, B, C, respectively,

CHAPTER II.

On Life Annuities in general.

§ 28. THE method of determining the present value of any annuity is, to find the present value of each year's rent as it becomes due; and the sum of all these will be the total present value of the annuity required. Such value will in all cases depend on the annual rate of interest concerned; and throughout the whole of the present work I have denoted this *annual* rate by ρ^* : consequently the *amount* of £1 at the end of a year will be denoted by $(1 + \rho)$; and the *present value* of £1 certain to be received at the end of 1, 2, 3, &c, years will be respectively denoted by $(1 + \rho)^{-1}$, $(1 + \rho)^{-2}$, $(1 + \rho)^{-3}$, &c.† The sum of which con-

* The *annual* rate should, in all cases of compound interest, be carefully distinguished from the *nominal* rate: but, such annual rate may always be expressed in terms of the nominal rate, as I have distinctly shown in another work. See *Doctrine of Interest and Annuities*, p. 16.

† As we shall often have occasion in the course of this treatise to find the present value of £1 certain to be received at the end of any number of years, and also the present value of £1 *per annum* for any number of years, at different rates of interest; I have (in order to avoid the necessity of referring to any other work) inserted at the end, two tables, which show the values here alluded to: being Tables LVIII and LIX, at the end of this work.

tinued to n terms, or $(1 + e)^{-1} + (1 + e)^{-2} + (1 + e)^{-3} + \dots + (1 + e)^{-n} = \frac{1 - (1 + e)^{-n}}{e}$, will denote the *present value of an annuity of £1 per annum for n years*: and if this series be continued to infinity, the sum of it, or $\frac{1}{e}$, will express the *present value of the perpetuity of the same annuity*. The principles, on which these observations are founded, have been fully explained in my treatise on the *Doctrine of Interest and Annuities*; but I have thought it necessary to mention them here, in order to prevent circumlocution in the investigation of the following problems.

§ 29. In life annuities, however, the rent of each year is to be received only on certain contingencies; consequently the present values above mentioned must be diminished in proportion to the probability of receiving them: and the sum of such values, for each successive year, will be the total present value of the life annuity required. Throughout the whole of this work I have supposed the annuity to be £1 per annum; in which case, the present value deduced will denote the *number of years purchase* that such annuity is worth: and which, being multiplied by any other annuity, will give the present value of such other annuity accordingly.

PROBLEM I.

§ 30*. To find the value† of an annuity granted upon any number of lives; that is, for as long as they shall all continue in being together.

SOLUTION.

Let A, B, C, &c, be the lives upon which the annuity is granted; and let the probability of each life continuing 1, 2, 3, &c, years, be as denoted in § 23: then it will follow, from what has been said in § 24, that the probability of all the lives continuing to the end of the first year will be $\frac{a' b' c' \&c}{a b c \&c}$: which, being

* This fundamental proposition, upon which the whole doctrine of annuities in a great measure depends, may be found in most authors who have treated on this subject. In the investigation of the subsequent Problems and their Corollaries, I shall refer to the similar propositions in the works of the five following authors: viz. Simpson's *Doctrine of Annuities and Reversions*, 1775; and his *Supplement* to the same, 1791. De Moivre's *Doctrine of Chances*, 3rd edit. 1756. Dodson's *Mathematical Repository*. Dr. Price's *Observations on Reversionary Payments*, 6th edit. 1803. Morgan's *Doctrine of Annuities and Assurances*, 1779: and his papers in the *Philosophical Transactions* for 1788, 1789, 1791, 1794, and 1800. Whereby the reader may the more readily compare them together, and judge of the respective merits of the rules which they have given for the solution of the same.

† By the term *value*, I mean the number of years purchase that the annuity is worth, agreeably to what I have just observed. This mode of expression will be used throughout the present work.

multiplied by $(1 + \rho)^{-1}$, or the present value of £1 certain to be received at the end of one year, will

produce $\frac{a' b' c' \mathcal{E}_c}{(1 + \rho) a b c \mathcal{E}_c}$ for the present value of the first

year's rent; or the *expectation** of receiving such sum on the contingency that all the lives continue to the end of the first year. In like manner, since the probability that all the lives will continue to the end

of the second year is $\frac{a'' b'' c'' \mathcal{E}_c}{a b c \mathcal{E}_c}$, if this expression be

multiplied by $(1 + \rho)^{-2}$, or the present value of £1 certain to be received at the end of two years, it will

produce $\frac{a'' b'' c'' \mathcal{E}_c}{(1 + \rho)^2 a b c \mathcal{E}_c}$ for the present value of the

second year's rent; or the *expectation* of receiving such sum on the contingency that all the lives will continue to the end of the second year. By the same method of reasoning it will be found that

$\frac{a''' b''' c''' \mathcal{E}_c}{(1 + \rho)^3 a b c \mathcal{E}_c}$ will denote the present value of the

third year's rent; or the *expectation* of receiving such sum on the contingency that all the lives will continue to the end of the third year. And in this manner

we must proceed for all the subsequent years of human life, the sum of all which terms†, or $\frac{1}{a b c \mathcal{E}_c} \times$

* See Ch. 1. § 5 and 6.

† When the sum of this series is to be determined in numbers, the terms of it must be carried to the extinction of the *oldest* life C, involved in the combination; at which period all the subsequent terms vanish because γ' , γ'' , &c, become equal to nothing. This observation will apply to all cases of combined lives.

$$\left[\frac{{}^i a {}^j b {}^k c \mathcal{E}_c}{(1+\rho)} + \frac{{}^{ii} a {}^{jj} b {}^{kk} c \mathcal{E}_c}{(1+\rho)^2} + \frac{{}^{iii} a {}^{jjj} b {}^{kkk} c \mathcal{E}_c}{(1+\rho)^3} + \dots \frac{\alpha \beta \gamma \mathcal{E}_c}{(1+\rho)^n} \right] \text{ will}$$

be the total present value of the annuity: where n in this case denotes the number of years between the age of the oldest of the given lives, and the age of the oldest life in the table of observations.

COROLLARY I.

§ 31. Now, when only one life A is concerned, this series will become $\frac{1}{a} \left[\frac{{}^i a}{(1+\rho)} + \frac{{}^{ii} a}{(1+\rho)^2} + \frac{{}^{iii} a}{(1+\rho)^3} + \&c \right]$ when two lives, A and B are concerned, it will become $\frac{1}{ab} \left[\frac{{}^i a {}^j b}{(1+\rho)} + \frac{{}^{ii} a {}^{jj} b}{(1+\rho)^2} + \frac{{}^{iii} a {}^{jjj} b}{(1+\rho)^3} + \&c \right]$. Hence, if we make the italic capitals $A, B, C, \&c$, to denote the value of an annuity on any single life A, B, C, &c; and $AB, AC, BC, \&c$, to denote the value of an annuity on any two joint lives AB, AC, BC, &c; and $ABC, \&c$, to denote the value of an annuity on any three joint lives ABC, &c: then in the case of a single life, we shall have

$$\begin{aligned} \frac{1}{a} \left[\frac{{}^i a}{(1+\rho)} + \frac{{}^{ii} a}{(1+\rho)^2} + \frac{{}^{iii} a}{(1+\rho)^3} + \&c \right] &= A \\ \frac{1}{b} \left[\frac{{}^j b}{(1+\rho)} + \frac{{}^{jj} b}{(1+\rho)^2} + \frac{{}^{jjj} b}{(1+\rho)^3} + \&c \right] &= B \\ \frac{1}{c} \left[\frac{{}^k c}{(1+\rho)} + \frac{{}^{kk} c}{(1+\rho)^2} + \frac{{}^{kkk} c}{(1+\rho)^3} + \&c \right] &= C \\ \&c \qquad \qquad \&c \qquad \qquad \&c \qquad \qquad \&c : \end{aligned}$$

and in the case of two joint lives, we shall have

$$\begin{aligned} \frac{1}{ab} \left[\frac{a^I b^I}{(1+\rho)} + \frac{a^{II} b^{II}}{(1+\rho)^2} + \frac{a^{III} b^{III}}{(1+\rho)^3} + \&c \right] &= AB \\ \frac{1}{ac} \left[\frac{a^I c^I}{(1+\rho)} + \frac{a^{II} c^{II}}{(1+\rho)^2} + \frac{a^{III} c^{III}}{(1+\rho)^3} + \&c \right] &= AC \\ \frac{1}{bc} \left[\frac{b^I c^I}{(1+\rho)} + \frac{b^{II} c^{II}}{(1+\rho)^2} + \frac{b^{III} c^{III}}{(1+\rho)^3} + \&c \right] &= BC \\ \&c \qquad \&c \qquad \&c \qquad \&c: \end{aligned}$$

and in the case of three joint lives, we shall have

$$\frac{1}{abc} \left[\frac{a^I b^I c^I}{(1+\rho)} + \frac{a^{II} b^{II} c^{II}}{(1+\rho)^2} + \frac{a^{III} b^{III} c^{III}}{(1+\rho)^3} + \&c \right] = ABC$$

&c &c &c &c.

COROLLARY II.

§ 32. Although there is no method of summing up these terms, or of abridging the general expression above given for finding the value of annuities on single and joint lives, but each respective term must be actually reduced to numbers, and the whole of them added together in order to determine the total value of the annuity; yet in finding the value of annuities on a number of single or joint lives, that is on lives of several successive ages, the process may be considerably abridged by deducing the value of an annuity on the next younger life from the value of an annuity on a life or lives each one year older. For, let ABC denote the value of an annuity on any number of joint lives A, B, C^* ; and ABC the value of an annuity

* In the investigation I shall confine the case to *three* joint lives: but it is easy to see that the method is general, *whatever* number be taken.

on the same number of joint lives each one year younger than A, B, C; and let $\overset{'}{a} \overset{'}{b} \overset{'}{c}$ &c be the number of persons living at the ages of those younger lives, as found in any table of observations: then, for

the very same reason that $\frac{1}{\overset{'}{a} \overset{'}{b} \overset{'}{c}} \left[\frac{\overset{'''}{a} \overset{'''}{b} \overset{'''}{c}}{(1+\rho)} + \frac{\overset{''''}{a} \overset{''''}{b} \overset{''''}{c}}{(1+\rho)^2} + \frac{\overset{'''''}{a} \overset{'''''}{b} \overset{'''''}{c}}{(1+\rho)^3} + \&c \right] = ABC$, we shall have $\frac{1}{\overset{'''}{a} \overset{'''}{b} \overset{'''}{c}} \left[\frac{\overset{''}{a} \overset{''}{b} \overset{''}{c}}{(1+\rho)} + \frac{\overset{'''}{a} \overset{'''}{b} \overset{'''}{c}}{(1+\rho)^2} + \frac{\overset{''''}{a} \overset{''''}{b} \overset{''''}{c}}{(1+\rho)^3} + \&c \right] = \overset{'''}{A} \overset{'''}{B} \overset{'''}{C}$. Wherefore, multiplying the first equation by $\overset{'''}{a} \overset{'''}{b} \overset{'''}{c}$, and the latter by $(1+\rho) \overset{'''}{a} \overset{'''}{b} \overset{'''}{c}$, we have

$$\frac{\overset{'''}{a} \overset{'''}{b} \overset{'''}{c}}{(1+\rho)} + \frac{\overset{''''}{a} \overset{''''}{b} \overset{''''}{c}}{(1+\rho)^2} + \&c = ABC \times \overset{'''}{a} \overset{'''}{b} \overset{'''}{c}$$

$$\frac{\overset{'''}{a} \overset{'''}{b} \overset{'''}{c}}{(1+\rho)} + \frac{\overset{''''}{a} \overset{''''}{b} \overset{''''}{c}}{(1+\rho)^2} + \&c = \overset{'''}{A} \overset{'''}{B} \overset{'''}{C} \times (1+\rho) \overset{'''}{a} \overset{'''}{b} \overset{'''}{c} - \overset{'''}{a} \overset{'''}{b} \overset{'''}{c}:$$

whence $\overset{'''}{A} \overset{'''}{B} \overset{'''}{C} (1+\rho) \overset{'''}{a} \overset{'''}{b} \overset{'''}{c} - \overset{'''}{a} \overset{'''}{b} \overset{'''}{c} = \overset{'''}{A} \overset{'''}{B} \overset{'''}{C} \times \overset{'''}{a} \overset{'''}{b} \overset{'''}{c}$.

$$\text{Consequently } \overset{'''}{A} \overset{'''}{B} \overset{'''}{C} = (1 + \overset{'''}{A} \overset{'''}{B} \overset{'''}{C}) \times \frac{\overset{'''}{a} \overset{'''}{b} \overset{'''}{c}}{\overset{'''}{a} \overset{'''}{b} \overset{'''}{c}} (1+\rho)^{-1}:$$

whence the following rule for finding the value of an annuity on any *single* life, the principle of which it is easy to apply to the case of any *joint* lives. Begin with the oldest life in the table of observations; add unity to the value of an annuity on that life (usually equal to 0) and multiply the sum by the expectation of a life one year younger receiving £1 at the end of a year; the product will be the value of an annuity on the life one year younger: this value being substituted for the value of an annuity on the oldest life, and the process repeated, will give the value of an

annuity on the next youngest life: and so on till we come to the age of the given life. Now, though the method of deducing the value of an annuity on any *given* life by means of this formula is rather more laborious than finding the numerical value of each term of the series given in the last corollary; yet the present formula has this advantage, that the several steps of the process give the values of annuities on *lives* of all the ages between the given life and the oldest life in the table of observations: whence, the calculation of the value of an annuity on lives of *all* those ages becomes scarcely more troublesome than the calculation of the value of an annuity on the youngest life.

§ 33. *Example 1.* Let it be required to find the value of an annuity on a life aged 90 years, allowing interest at the rate of 5 per cent per annum, and according to the probabilities of life in general as observed in *Sweden*.

The value of an annuity on a life aged 96 is evidently equal to 0:

$$\text{aged 95,} = (1 + 0) \times \frac{1}{2} \times .9524 = .4762$$

$$94, = (1 + .4762) \times \frac{2}{5} \times .9524 = .5624$$

$$93, = (1 + .5624) \times \frac{5}{11} \times .9524 = .6764$$

$$92, = (1 + .6764) \times \frac{11}{21} \times .9524 = .8363$$

$$91, = (1 + .8363) \times \frac{21}{33} \times .9524 = 1.1129$$

$$90, = (1 + 1.1129) \times \frac{33}{47} \times .9524 = 1.4129$$

whence it appears that the value required is equal to 1.4129*.

§ 34. *Example 2.* What is the value of an annuity on the joint lives of a man aged 90, and a woman aged 84; reckoning interest at 4 per cent and the probabilities of living, among males and females respectively, as observed in Sweden?

By beginning with the oldest life in the table of observations, joined to another whose difference of age is 6 years, it will be found that the value of an annuity on two similar joint lives aged 95 and 89 is equal to 0:

$$94 \text{ and } 88 = (1+0) \times \frac{1 \times 76}{4 \times 99} \times .9615 = .1845$$

$$93 \text{ and } 87 = (1+.1845) \times \frac{4 \times 99}{10 \times 129} \times .9615 = .3496$$

$$92 \text{ and } 86 = (1+.3496) \times \frac{10 \times 129}{17 \times 169} \times .9615 = .5827$$

$$91 \text{ and } 85 = (1+.5827) \times \frac{17 \times 169}{26 \times 224} \times .9615 = .7507$$

$$90 \text{ and } 84 = (1+.7507) \times \frac{26 \times 224}{38 \times 299} \times .9615 = .8629$$

Whence, the value of the annuity required is equal to .8629.†

* Dr. Price has given *incorrect* values for annuities on *lives in general*, according to the table of observations for Sweden. See *Obs. on Rev. Pay.* vol. ii, p. 422. He has taken a *mean* between the value of annuities on male and female lives, which is evidently *erroneous*.

† In Tables XVII to XXIV at the end of this work, the values are deduced from the probabilities of living amongst males and

§ 35. But the best mode of computing the values of annuities is by means of *logarithms*: and the method of doing this will be sufficiently evident from an inspection of the formula in page 31, and from the principles which have been just laid down. For, since $ABC = (1 + ABC) \times \frac{a b c}{a b c} (1 + e)^{-1}$, it is manifest that $\text{Log. } ABC = \text{Log. } a + \text{Log. } b + \text{Log. } c + \text{Log. } (1 + e)^{-1} + \text{Log. } (1 + ABC) - [\text{Log. } a + \text{Log. } b + \text{Log. } c]$: whence the calculations, which appear so laborious and intricate when 2 or 3 joint lives are involved, are reduced to the simple operations of addition and subtraction.

§ 36. In calculating the values of annuities according to this method, the following directions should be observed. Begin with the oldest life, C; and write down horizontally on a paper (divided into columns as in the annexed specimen) the logarithms of the number of persons living at the end of n , $(n - 1)$, $(n - 2)$, &c, years from birth; n denoting the number of years between *birth* and the age of the oldest life in the table of observations. Then, proceeding to the next oldest life B, write down in a similar manner, under the former, the logarithms of the number of persons living

females collectively: but it would be very desirable to have tables of the values of annuities on joint lives, where the ages of the two sexes are kept *distinct* from each other. See the note to the second example in Question IV, Chapter XII.

at the end of $(n - \Delta)$, $(n - \Delta - 1)$, $(n - \Delta - 2)$, &c, years from birth: Δ denoting the difference of age between B and C. And so on to the next oldest life, according to the number of joint lives required. Add these several perpendicular values together, and write down under each of them the logarithm of $(1 + \rho)^{-1}$. This being done, the subsequent operations become extremely easy, as will sufficiently appear from the following specimen, which shows the method of obtaining the values of annuities on two joint lives of all ages whose difference of age is 10 years; reckoning interest at 4 per cent, and the probabilities of living as at *Northampton*.

Specimen of an easy and expeditious method of calculating the values of two joint lives.
Difference of age 10 years—Interest 4 per cent—Northampton Observations.

	Index to the values in each line.	Ages 95 and 85	Ages 94 and 84	Ages 93 and 83	Ages 92 and 82	Ages 91 and 81	Ages 90 and 80
D	Log ^a . of 1, 4, 9, 16, 24, 34, &c.	.0000000	.6020600	.9542425	.2041200	.3802112	.5314789
E	Log ^a . of 145, 186, 234, 289, 346, 406, &c.	.1613680	.2695129	.3692159	.4608978	.5390761	.6065260
F	Sum of the two preceding logarithms	.1613680	.8715729	.3234584	.6650178	.9192873	.1400049
G	Log. of (1.04) ⁻¹ = Log. of .961538	.9829667	.9829667	.9829667	.9829667	.9829667	.9829667
H	Log ^a . of the Nat. numbers in S, added to unity		.0745970	.1471660	.2080568	.2705572	.3177416
I	Sum of the logarithms in F, G, H, ...	.1443347	.9291366	.4535011	.8560413	.1728112	.4407142
K	Log ^a . of F in the subsequent column	.8715729	.3234584	.6650178	.9192873	.1400049	&c
L	Difference of the two preceding logarithms.	.2727618	.6056782	.7885733	.9367540	.9328063	&c
M	Log ^a . of (1.04) ⁻² (1.04) ⁻³ (1.04) ⁻⁴ &c	.5521661	.5691994	.5862328	.6032661	.6202995	&c
N	Sum of the logarithms in K and M	.4237390	.8926578	.2512506	.5225534	.7668044	&c
O	Sum of the logarithms in L and N: $\left\{ \begin{array}{l} \text{or in I and M} \end{array} \right\}$. . .	.6965008	.4983360	.0398239	.4593074	.7931107	&c
P	Nat. numbers to the logarithms in N	26.53011	78.10121	178.3407	333.0837	575.8434	&c
Q	Nat. numbers to the logarithms in O	4.97165	31.50185	109.6032	287.9436	621.0271	&c
R	Sums of the Nat. numbers in P and Q	31.50176	109.60306	287.9439	621.0273	1196.8705	&c
S	Nat. numbers to the logarithms in L, $\left\{ \begin{array}{l} \text{which are the values of the an. required} \end{array} \right\}$. . .	.18740	.40335	.61457	.86448	1.07846	&c.

§ 37. In taking out these logarithms from the Tables, the *decimal* parts only are to be retained : and the same observation will apply to the addition and subtraction of the logarithms in the specimen. No error is likely to arise from thus omitting the index, because in the last logarithms resulting from the process (and which are the logarithms of the values of the annuity required in line L) the index when positive never exceeds 1 ; and when negative it cannot well be mistaken.

In every column, the numbers in the lines D, E, F, G, H, I, K, L, show the process for finding the *value* of the annuity : and are all that are required for *that* purpose. The last line L shows the logarithms of such values ; and the natural numbers corresponding to those logarithms (that is, the true values of the annuity required) are set down in the line S. The reason for their being thus kept detached will appear in the sequel ; when I come to notice the subsequent lines M, N, O, P, Q, R : and which, for the present, we must suppose not to be in existence.

§ 38. This specimen may be accommodated to the calculation of the values of annuities on *single* lives, by striking out the logarithms in the line E and F. Or, it may be accommodated to the calculation of the values of annuities on any *three* joint lives ABC, if to the logarithms in the lines D and E we add (in order to obtain the line F) the logarithms of the number of persons living at the end of $(n - \delta)$,

$(n - \delta - 1)$, $(n - \delta - 2)$, &c years from birth: δ denoting the difference between the age of A and C. Hence it is evident that, by this method, the computations of the values of annuities on any *two*, *three*, or *four* joint lives, are nearly as easy as the computations of the values of annuities on any *single* lives: and that the principal labour will consist in taking out the logarithms of the number of persons living at the respective ages; and in finding the natural numbers corresponding to the values in the line L.

§ 39. It is obvious that an error in any one of the numbers will cause an error in *all* the subsequent numbers: and therefore, since the detection of an error at the end of the process might be attended with infinite trouble, it would be a great satisfaction to be enabled to prove the accuracy of the numbers in *each* column, as we go on. A method of doing this has been pointed out by Mr. Morgan, in his *Doctrine of Annuities, &c*, page 58; from which I have deduced the following rule.

§ 40. In the horizontal line M write down the logarithms of $(1 + \rho)^{-(n-\delta-1)}$, $(1 + \rho)^{-(n-\delta-2)}$, $(1 + \rho)^{-(n-\delta-3)}$, &c*: δ denoting the difference of

* The best and most expeditious method of finding the subsequent logarithms of these quantities, after having obtained the logarithm of $(1 + \rho)^{-(n-\delta-1)}$, is to add thereto the logarithm

age between the oldest and the youngest life involved. Add these to the logarithms in the line K and set down the sum of them in the line N: add them also to the logarithms in the line I, and set down the sum of these in the line O. Next take out the natural numbers* to the logarithms thus set down in the lines N and O; and place these in the lines P and Q respectively. Then if the natural number in the line Q, in any column, is equal to the sum of the natural numbers in the lines P and Q, in the preceding column, all the values *thus far* have been correctly determined. But, if the first 5 or 6 figures of such number in Q do not agree with the first 5 or 6 figures of the number in the line R, in the preceding column, it will show that there is some inaccuracy in the numbers employed *since the last proof*: and consequently, a revision of the process *from that point* becomes necessary.†

of $(1 + p)$ for each succeeding year. Thus, in the preceding specimen, the logarithm of $(1 + p)^{-(n-3-1)}$, or $(1.04)^{-3}$, is equal to .5521661; and the logarithm of 1.04 is equal to .0170333: therefore .0170333 added continually to .5521661, will give all the subsequent values in the line M.

* In order to take out these logarithms in a proper manner, it might seem necessary to retain the *indices* of the logarithms throughout the whole process: but, after a short trial, the computist will find that he is not likely to be led into any error by omitting them, and that it will save much time and labour if he reject them altogether.

† In revising the process here alluded to, it will be necessary to examine the numbers in the lines N, P, R, S, in the *preceding*

§ 41. Now, though the labour of computing the values of annuities on any lives is somewhat increased by this method of proceeding, yet the additional trouble is more than compensated by the satisfaction which results from a knowledge of the accuracy of the values obtained; as well as by the time and labour which are saved in consequence of detecting every error as it arises.

§ 42. I have entered thus fully into the best method of deducing the values of annuities on single and joint lives, because we are still in want of many accurate tables upon this subject, not only to complete those we are already possessed of; but more particularly some *new* ones founded on the probabilities of life in *this* country as observed amongst annuitants *only*, and also on the probabilities of life as observed in the kingdom *at large*. Such tables would be extremely valuable for general use; but the trouble of calculating them, even with the aid of logarithms, is

column; the numbers in the lines D, E, F, in the *subsequent* column; and the numbers in the lines G, H, I, K, L, M, O, Q, in the column where the proof is found to fail.

Dr. Price observes (*Obs. on Rev. Pay.* vol. i. p. 218) that in calculating the *last* value (that is, when we come to the age of the youngest life in the table of observations) this proof will change into a *new* proof, verifying all the preceding values. This, however, is not the case: the rule which he has given for this purpose is still the *same* proof differently worded; and, *in common with* each proof in its respective column, verifies all the preceding values.

so great that few *individuals* (were they possessed of sufficient *data*, whereon to proceed) would be disposed to undertake so laborious a task: and it unfortunately happens that those *Societies*, whose province it seems more peculiarly to be to patronize an undertaking of that kind, would find little interest in giving the public any additional information on this subject.

§ 43. At the end of the present work I have inserted a collection of all the tables of the values of annuities on single and joint lives, that have hitherto been published*: from which, the computist may select such as appear to relate more particularly to the cases laid before him. Or, he may occasionally make use of them all, according to the condition, situation, and circumstances of the parties concerned. The use and application of these tables will be more fully explained in the last chapter.

COROLLARY III.†

§ 44. By means of the general expression in the problem, for finding the value of an annuity on any

* Tables V to XIII are (by permission of Baron Maseres) taken from his valuable *Treatise on Annuities*: Tables XVI to XXIV, and from XXVII to XLIV are taken from Dr. Price's *Observations on Rev. Pay.* vol. ii. edit. 1803: Tables XLVII and XLVIII are taken from Mr. Simpson's *Treatise on Annuities*. Table XLIX is taken from Mr. Dodson's *Mathematical Repository*, vol. ii: and Table L is taken from Dr. Price's work above mentioned.

† Price, Note (B). Simpson's Sup. Prob. 13.

given lives, we may determine the value of a *Deferred* life annuity; that is, of an annuity which is not to commence till the end of a given number of years ($=n$) provided the given lives are then in being, and to continue from that period to the extinction of such given lives. For, let A, B, C, &c, be the given lives; and let $\alpha, \alpha', \alpha'', \alpha''', \&c$; $\beta, \beta', \beta'', \beta''', \&c$; $\gamma, \gamma', \gamma'', \gamma''', \&c$; &c denote the number of persons living, according to any table of observations, at the end of $n, (n+1), (n+2), (n+3), \&c$, years, from the several ages of A, B, C, &c, respectively, as explained in § 23: then will the value of an annuity on the lives ABC&c, be expressed by the following series,

$$\frac{1}{abc \&c} \times \left[\frac{\alpha' \beta' \gamma' \&c}{(1+\rho)} + \frac{\alpha'' \beta'' \gamma'' \&c}{(1+\rho)^2} + \frac{\alpha''' \beta''' \gamma''' \&c}{(1+\rho)^3} + \dots + \frac{\alpha' \beta' \gamma' \&c}{(1+\rho)^{n+1}} + \frac{\alpha'' \beta'' \gamma'' \&c}{(1+\rho)^{n+2}} + \frac{\alpha''' \beta''' \gamma''' \&c}{(1+\rho)^{n+3}} + \&c \right];$$

where the first part is to be continued to n terms, and where the latter part (or that which is deferred) is to be continued to the utmost extent of human life. Consequently this latter

part, or the series $\frac{1}{abc \&c} \times \left[\frac{\alpha' \beta' \gamma' \&c}{(1+\rho)^{n+1}} + \frac{\alpha'' \beta'' \gamma'' \&c}{(1+\rho)^{n+2}} + \frac{\alpha''' \beta''' \gamma''' \&c}{(1+\rho)^{n+3}} + \&c \right]$ will express the value of an annuity to be entered on* at the end of n years; and the sum

* The first payment of which, however, will not take place till the end of the $(n+1)^{st}$ year. See my *Doctrine of Interest and Annuities*, p. 65, note.

of which I shall denote by $(ABC\&c)^d$.^{*} Now, if we have tables of the values of annuities on single and joint lives of all ages, we may easily deduce the value of this series (without the actual calculation of each separate term) from *the value of an annuity on the same number of lives each n years older than the given lives*. For, if this latter value be denoted by $A'B'C'\&c$,[†] we shall have, according to the prin-

* Care should be taken not to mistake the new character, here introduced, for the *index* of the quantity within the vinculum. In the doctrine of annuities, the notation is, in several instances, different from that of the common analysis of quantities: for instance, the juxtaposition of the letters $ABC\&c$, does not denote the multiplication of the quantities represented by those letters, because those letters jointly represent but one quantity; neither does A^x , as used by Mr. Morgan, denote the x th power of an annuity on the life of A. These new methods of notation, after a proper explanation, do not cause any ambiguity; as will be readily felt by those who are acquainted with the Differential and Integral Calculus: and with respect to the symbol here introduced, I would observe that (similar to the like character in the Differential Calculus) it does not altogether represent a quantity, but merely an *operation*. This operation consists in substituting, for the value of an annuity on the given lives within the vinculum, the value of an annuity on the same number of lives each older by the given term, and in multiplying the same by the expectation that the original lives will receive £1 at the end of that term.

† I would here observe that I denote the value of an annuity on a life n years older than A, B, C, &c, by the characters A' , B' , C' , &c: and on two joint lives, each n years older than AB, AC, BC, &c; by the characters $A'B'$, $A'C'$, $B'C'$, &c: and on three joint lives, each n years older than ABC, &c, by $A'B'C'$, &c.

ciples laid down in the problem, $ABC&c = \frac{1}{a\beta\gamma\wp c} \times \left[\frac{a'\beta'\gamma'\wp c}{(1+\wp)} + \frac{a''\beta''\gamma''\wp c}{(1+\wp)^2} + \frac{a'''\beta'''\gamma'''\wp c}{(1+\wp)^3} + \&c \right]$: therefore, multiplying both sides of this equation by $\frac{a\beta\gamma\wp c}{abc\wp c} (1+\wp)^{-n}$, we produce $ABC&c \times \frac{a\beta\gamma\wp c}{abc\wp c} (1+\wp)^{-n} = (ABC&c)^d$: whence the following general rule.

§ 45. *Find the value of an annuity on the same number of lives, each as many years older than the given lives as are equal to the number of years during which the annuity is deferred; find also the expectation* of the given lives receiving £1 at the end of that term: the product of these two quantities will be the value required.†*

* The expectation of receiving any sum of money at the end of any given term is equal to the present value of such sum multiplied by the probability that the given lives will continue in being so long: that is, equal to $\frac{a\beta\gamma\wp c}{abc\wp c} \times (1+\wp)^{-n}$. See Chap. I, § 5 and 6.

† The periods for which any annuity may be deferred, and the variety of ages with which the same may be combined, would render it an endless labour to compute a table of the value of deferred life annuities to suit every case. Mr. Baron Maseres has given a partial table of this kind, which shows the present value of annuities, on single lives, to commence at the end of *thirty* years, adapted to every age from 3 to 64 years, founded on M. De Parcieux's tables of observations, and at the several rates of $3\frac{1}{2}$, 4, $4\frac{1}{2}$, and 5 per cent interest. See his *Doctrine of Annuities*, p. 283 to 287.

For examples of the use and application of this corollary, see Question VI in Chapter XII.

COROLLARY IV.*

§ 46. By the help of the series in the last corollary we may determine the value of a *Temporary* life annuity; that is, of an annuity which is to commence immediately, but to continue only for a given number of years ($=n$) which is less than that to which it is possible that the given life or lives may extend†, and then to cease. For, supposing every thing to remain as in the last corollary, it is evident that the first n

terms of the series there given, or $\frac{1}{abc\&c} \times \left[\frac{a'b'c'\&c}{(1+\rho)} + \frac{a''b''c''\&c}{(1+\rho)^2} + \frac{a''b''c''\&c}{(1+\rho)^3} + \dots \dots \dots \frac{\alpha\beta\gamma\&c}{(1+\rho)^n} \right]$ will be the value of the temporary annuity required, and which I shall denote by $(ABC\&c)^{t,\dagger}$. Now by means of

* Simpson's Sup. Prob. 6. De Moivre, Prob. 26. Morgan, Prob. 4. Dodson, vol. iii. Ques. 2, 7, and 10.

† This is always understood in cases of this kind: and it may be here useful to remark that the term during which it is possible any given life may be prolonged is equal to the difference between the age of that life, and the age of the oldest life in the table of observations. In the case of joint lives, it is equal to the difference between the oldest of such joint lives, and the age of the oldest life in the table of observations.

‡ Agreeably to what has been said in the note in page 43, the reader must be careful not to mistake this character for the *index*

the tables above mentioned, we may easily deduce the value of this series without the actual enumeration of its several terms: for, it is evident on inspection that the value of the first n terms of the series in the last corollary is equal to the value of the whole series *minus* the sum of all the terms after the n^{th} term. But the sum of all the terms after the n^{th} term has been found, by the last corollary, to be equal to $(ABC\&c)^d$: consequently the sum of the first n terms is equal to $ABC\&c - (ABC\&c)^d$; whence the following rule.

§ 47. *Find the value of an annuity on the given lives deferred during the given period, and deduct this value from the value of an annuity on the given lives: the difference will be the value required.*

For examples of the use and application of this corollary, see Question VII in Chapter XII.

COROLLARY V.

§ 48. By a similar method of reasoning we may find the value of a *Deferred and Temporary* life annuity; that is, of an annuity which is not to commence till the end of a given number of years ($=n$),

of the quantity within the vinculum. For, the symbol here introduced denotes also rather an *operation* to be performed; and which operation consists in substituting, for the original quantity, the *difference* between such quantity and the new one found by the method laid down in the note above alluded to.

and then to continue only during another period ($= \nu$), which is less than that to which it is possible the given lives may then be prolonged. And, from what has been said in the third corollary, the truth of the following rule will be evident.

§ 49. *From the value of an annuity on the given lives, deferred for the term n , subtract the value of the same annuity on such lives, deferred for the term $(n + \nu)$: the difference will be the sum of the required ν terms of the series, or the value of the annuity proposed.*

COROLLARY VI.*

§ 50. If the annuity is to be enjoyed for a term certain ($= n$), and after that during the continuance of any given lives, its value will be expressed by the following series $\frac{1}{(1+\rho)} + \frac{1}{(1+\rho)^2} + \frac{1}{(1+\rho)^3} + \dots + \frac{1}{(1+\rho)^n} +$

$$\frac{1}{\rho} \times \left[\frac{a' \beta' \gamma' \&c}{(1+\rho)^{n+1}} + \frac{a'' \beta'' \gamma'' \&c}{(1+\rho)^{n+2}} + \frac{a''' \beta''' \gamma''' \&c}{(1+\rho)^{n+3}} + \&c \right].$$

But, the first n terms of this series, or $\frac{1}{(1+\rho)} + \frac{1}{(1+\rho)^2} + \frac{1}{(1+\rho)^3} + \dots + \frac{1}{(1+\rho)^n}$, are equal to $\frac{1-(1+\rho)^{-n}}{\rho}$, or

to the present value of an annuity of £1 per annum for n years; and the remaining part of the series is,

* Morgan, Prob. 15.

by the third corollary, equal to $(ABC&c)^d$: whence the following rule.

§ 51. *To the present value of an annuity certain for the given term, add the value of an annuity on the given lives deferred for that term: the sum will be the answer required.*

For examples of the use and application of this corollary, see Question XXII in Chapter XII.

COROLLARY VII.

§ 52. If, in the general expression for the value of an annuity on a single life A , we suppose ρ to vanish, or that money does not bear any interest; then will such expression be reduced to the series $\frac{1}{a}(a' + a'' + a''' + \&c)$, and which is the value of an annuity on the life A considered as a *yearly annuitant* without interest of money: whence the following rule. Divide the sum of all the living, at every age after the age of the given life, by the number of persons living at that age; and the quotient will be the value required.

In a subsequent chapter I shall show that we must add $\frac{1}{4}$ or $\frac{1}{8}$ to the value thus found, in order to obtain the value of a similar annuity on the life A considered as a *half yearly* or *quarterly annuitant**; but when

* See Chapter X.

the life is considered as a *momently* annuitant, we must add $\frac{1}{2}$ to such value. In this last case the required value coincides with the sums of the probabilities that such life will attain to the end of the 1st, 2nd, 3rd, &c, *moments* from the present time to the end of its possible existence: and is the same with what is denominated, by writers on this subject, the *Expectation of Life*; or the number of years which, taking lives of the same age one with another, any one of those lives may be considered as sure of enjoying; those who live *beyond* that period, enjoying as much *more* in proportion to their number, as those who *fall short* of it enjoy *less*. Consequently the rule for finding the Expectation of Life will be as follows.

§ 53. *Divide the sum of all the living, at every age after the age of the given life, by the number of persons living at that age: half unity added to the quotient will be the value required.*

§ 54. As we may sometimes have occasion to determine this value in order to compare the probabilities of life, or the values of annuities, according to different tables of observations, as well as for various purposes of approximation to be explained hereafter, I have here inserted the rule for finding such values according to any table of observations. But in Table II at the end of the present treatise, I have given the Expectation of Life for every *fifth* year as deduced from the several tables of observations mentioned in page 9: and in

Tables IV, XV, XXVI and XLVI, I have given the Expectations of Life for *every* year, as deduced from the tables of observations immediately preceding*.

PROBLEM II.†

§ 55. To find the value of an annuity granted upon the longest of any number of lives; that is, for as long as any one of them is in existence.

SOLUTION.

Let A, B, C, &c, be the lives upon which the annuity is granted, and let the probability of each life continuing 1, 2, 3, &c, years, be as denoted in § 23: then, the probability that some one or other of these will live to the end of the first year will (by § 26) be expressed by $1 - (1 - \frac{a'}{a}) \times (1 - \frac{b'}{b}) \times (1 - \frac{c'}{c}) \times \&c$
 $= \frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c} + \&c - \frac{a'b'}{ab} - \frac{a'c'}{ac} - \frac{b'c'}{bc} - \&c + \frac{a'b'c'}{abc} +$
 &c; which, being multiplied by $(1 + e)^{-1}$, or the present value of £1 certain to be received at the end of one year, will give the present value of the first year's rent, or the *expectation* of receiving such sum on the

* I would here observe that, according to the hypothesis of M. De Moivre, which will be explained hereafter, the expectation of any single life is equal to *half the complement* of that life: consequently the Complement of any life is equal to twice the Expectation of that life. This mode of expression is sometimes used, even when speaking of values *deduced from real observations*.

† Simpson, Prob. 2. De Moivre, Prob. 4 and 5. Dodson, vol. ii. Ques. 76 to 86: vol. iii. Ques. 14.

contingency that any one of the lives continues to the end of the first year. In like manner, since the probability that some one or other of the lives will live to the end of the second year is $1 - (1 - \frac{a''}{a}) \times (1 - \frac{b''}{b})$

$$\times (1 - \frac{c''}{c}) \times \&c = \frac{a''}{a} + \frac{b''}{b} + \frac{c''}{c} + \&c - \frac{a''b''}{ab} - \frac{a''c''}{ac} - \frac{b''c''}{bc} \\ - \&c + \frac{a''b''c''}{abc} + \&c; \text{ it follows that if this expression}$$

be multiplied by $(1 + \rho)^{-2}$, or the present value of £1 certain to be received at the end of two years, it will give the present value of the second year's rent, or the *expectation* of receiving such sum on the contingency that some one or other of the lives will live to the end of the second year. By the same method of reason-

$$\text{ing it will be found that } 1 - (1 - \frac{a'''}{a}) \times (1 - \frac{b'''}{b}) \times \\ (1 - \frac{c'''}{c}) \times \&c = \frac{a'''}{a} + \frac{b'''}{b} + \frac{c'''}{c} + \&c - \frac{a'''b'''}{ab} - \frac{a'''c'''}{ac} - \frac{b'''c'''}{bc} \\ - \&c + \frac{a'''b'''c'''}{abc} + \&c, \text{ being multiplied by } (1 + \rho)^{-3},$$

will give the present value of the third year's rent, or the *expectation* of receiving such sum on the contingency that some one or other of the lives will live to the end of the third year. And so on for all the subsequent years, to the utmost extent of human life: the sum of all which expectations,* or the series

* When the ages of the given lives differ, the number of terms composing those several series will also differ: but, I would here observe, once for all, that in every case of combined lives the terms must be continued to the extinction of the oldest life in-

$$\begin{aligned}
& \frac{1}{(1+p)} \left[\frac{a}{a} + \frac{b}{b} + \frac{c}{c} + \&c - \frac{a'b}{ab} - \frac{a'c}{ac} - \frac{b'c}{bc} - \&c + \frac{a'b'c}{abc} + \&c \right] + \\
& \frac{1}{(1+p)^2} \left[\frac{a}{a} + \frac{b}{b} + \frac{c}{c} + \&c - \frac{a''b}{ab} - \frac{a''c}{ac} - \frac{b''c}{bc} - \&c + \frac{a''b''c}{abc} + \&c \right] + \\
& \frac{1}{(1+p)^3} \left[\frac{a}{a} + \frac{b}{b} + \frac{c}{c} + \&c - \frac{a'''b}{ab} - \frac{a'''c}{ac} - \frac{b'''c}{bc} - \&c + \frac{a'''b'''c}{abc} + \&c \right] +
\end{aligned}$$

will be the total present value of the annuity required.

§ 56. But, the first collateral column in this general expression denotes the value of an annuity on the life A; the second denotes the value of an annuity on the life B; the third denotes the value of an annuity on the life C; &c. In like manner, the fourth, fifth and sixth collateral columns denote the values of an annuity on the joint lives AB, AC, BC, &c respectively: and so on. Whence, if we substitute for these respective values the characters mentioned in Prob. I, cor. 1, the whole expression, or the sum of all the terms in the above series, will become equal to $A + B + C + \&c - AB - AC - BC - \&c + ABC + \&c$: consequently, the value of an annuity, to continue as long as any one of the given lives is in existence, is equal to the sum of the values of an annuity on all the single lives, *minus* the sum of the values of an annuity on all the joint lives combined two and two,

resolved in the combination; or universally, *each* series must be continued till the terms vanish, which will happen when either of the lives involved in such series has arrived at the extremity of human life.

plus the sum of the values of an annuity on all the joint lives combined three and three, minus the sum of the values of an annuity on all the joint lives combined four and four; and so on. Therefore, when the values of an annuity on the single and joint lives are given, the value of an annuity on the longest life may be very easily determined: and, for the sake of a more convenient reference, I shall take L to denote the value found by this rule; the number of lives, which it is intended to represent, being always explained when the character is used.

For examples of the use and application of this problem, see Questions VIII and IX in Chapter XII.

COROLLARY I.*

§ 57. If the lives are all equal, or of the same age A , and their number be represented by n ; then the probability that some one or other of such lives will continue to the end of the first, second, third, &c. years respectively, will be severally denoted by $1 - \left(1 - \frac{1}{a}\right)^n$, $1 - \left(1 - \frac{1}{a}\right)^n$, $1 - \left(1 - \frac{1}{a}\right)^n$, &c. Let these quantities be severally expanded by means of the binomial theorem, in order to reduce them to simple terms, and then be respectively multiplied into the present value of £1 due at the end of those years; the sum of them, or the series

* Dodson, vol. ii. Cases: 78, 85, and 86:—Vol. iii. Cases: 11:

$$\begin{aligned} & \frac{1}{(1+g)} \left[n \cdot \frac{a}{a} - \frac{n(n-1)}{2} \cdot \frac{a'a}{aa} + \frac{n(n-1)}{2} \cdot \frac{n-2}{3} \cdot \frac{a'a'a}{aaa} - \&c \right] + \\ & \frac{1}{(1+g)^2} \left[n \cdot \frac{a}{a} - \frac{n(n-1)}{2} \cdot \frac{a''a}{aa} + \frac{n(n-1)}{2} \cdot \frac{n-2}{3} \cdot \frac{a''a'a}{aaa} - \&c \right] + \\ & \frac{1}{(1+g)^3} \left[n \cdot \frac{a}{a} - \frac{n(n-1)}{2} \cdot \frac{a'''a}{aa} + \frac{n(n-1)}{2} \cdot \frac{n-2}{3} \cdot \frac{a'''a'a}{aaa} - \&c \right] + \end{aligned}$$

will be the total present value of the annuity required.

§ 58. Whence (if we take AA , AAA , &c, to denote the value of an annuity on two, three, &c, equal joint lives) it is manifest that the value of an annuity on the longest of any number of lives, all equal to A , will be $nA - \frac{n(n-1)}{2} AA + \frac{n(n-1)}{2} \cdot \frac{n-2}{3} AAA - \&c$.

Therefore, if the number of lives be two, this expression will become $2A - AA$; if three, it will become $3A - 3AA + AAA$; if four, it will become $4A - 6AA + 4AAA - AAAA$; and so on.

For examples of the use and application of this corollary, see Question VIII in Chapter XII.

COROLLARY II.*

§ 59. If we wish to determine the value of an annuity on the longest of any number of lives†, *Deferred*

* Dodson, vol. iii. Ques. 13 and 15.

† The investigations in the present corollary are confined to the case of three lives: but it is easy to apply the same method of reasoning, *whatever* be the number of lives involved.

for a given term ($= n$), it will be evident (from what has been said in Prob. I, cor. 3) that the several perpendicular series in page 52 must not commence till after the n^{th} term; and thence be continued to the utmost extent of human life. Consequently, the required value of such deferred annuity will be equal to $(A)^d + (B)^d + (C)^d - (AB)^d - (AC)^d - (BC)^d + (ABC)^d$: an expression which I shall, for the sake of a more convenient reference, denote by $(L)^d$. Now, if for each of these several quantities we substitute its corresponding value, agreeably to the principles laid down in page 44, the formula denoted by $(L)^d$ will become equal to $A \times \frac{a}{a} (1 + \rho)^{-n} + B \times \frac{\beta}{b} (1 + \rho)^{-n} + C \times \frac{\gamma}{c} (1 + \rho)^{-n} - AB \times \frac{a\beta}{ab} (1 + \rho)^{-n} - AC \times \frac{a\gamma}{ac} (1 + \rho)^{-n} - BC \times \frac{\beta\gamma}{bc} (1 + \rho)^{-n} + ABC \times \frac{a\beta\gamma}{abc} (1 + \rho)^{-n}$: which is an expression more convenient for practice; and from either of which we deduce the following rule.

§ 60. *Substitute the values of deferred annuities on the single and joint lives, in the general rule in the problem, instead of the values of annuities on the whole continuance of those lives; and proceed with these substituted values according to the directions given in such rule: the result will be the answer required.*

For examples of the use and application of this corollary, see Question XI in Chapter XII.

COROLLARY III.

§ 61. If an annuity on the longest of any number of lives, deferred for any given term, depends on the joint continuance of all those lives to the end of that term; the value of it will be equal to $L' \times \frac{a \beta \gamma}{a \beta \gamma} (1 + e)^{-n}$: that is, equal to the value of an annuity on the longest of the same number of lives, each older by the given term than the given lives,* multiplied by the expectation that the joint lives shall receive £1 at the end of that term: and this case must be carefully distinguished from the one mentioned in the preceding corollary.

* Upon the same principle that I have taken $(ABC^t c)^d$ and $(ABC^t c)^t$, in Prob. I, cor. 3 and 4, to denote the value of a deferred and temporary annuity on the joint lives ABC &c, I shall take $(L)^d$ and $(L)^t$ as the expressions for the value of a deferred and temporary annuity on the longest of those lives. Having also taken $A^t B^t C^t c$ to denote the value of an annuity on the same number of joint lives, each older by the given term than the lives ABC &c, I shall in like manner take L' as the expression for the value of a similar annuity on the longest of such lives. This method of notation will be found very convenient in many of the subsequent problems: and, being very simple in its principle, will not cause any confusion or intricacy from its novelty. The number of lives denoted by L will be always mentioned whenever the character is used: as I have already observed in page 53.

For examples of the use and application of this corollary, see the Scholium to Question XI in Chapter XII.

COROLLARY IV.

§ 62. Having found, by means of the second corollary here given, the value of a deferred annuity on the longest of any number of lives, we may easily determine the value of a *Temporary** annuity on the longest of such lives: it being nothing more than the difference between the value of such deferred annuity and the value of a similar annuity on the whole continuance of the lives, as found by the problem. That is, the formula $L - (L)^d$ will in all cases denote the value of such temporary annuity: similar to what takes place with respect to joint lives, as already explained in page 46.

For examples of the use and application of this corollary, see Question XII in Chapter XII.

* That is, of an annuity which is to commence immediately, but to continue only during a given number of years ($= n$) which is less than that to which it is possible *either* of the given lives may extend. Therefore such term must not be greater than the difference between the age of the *youngest* life involved, and the age of the *eldest* life in the table of observations; for, in such case, the value is found by the Problem. See what has been said respecting single and joint lives in the note in page 45.

PROBLEM III.*

§ 63. To find the value of an annuity granted upon any number of lives, but to continue only as long as any number ($= n$) of them are in being together.

SOLUTION.

Let the lives on which the annuity is granted be A, B, C, &c: and the probability of each life continuing 1, 2, 3, &c, years be as denoted in § 23. Now if we confine this case to that of an annuity granted upon three lives, and to continue as long as any *two* of them (viz. AB, AC, or BC) are in being together†, it is evident that the chance of the annuity being received in any one year will depend upon either of these four different events: 1° that all the lives continue in being together to the end of that year, the probability of which in the first year is $\frac{a' b' c'}{a b c}$: 2° that A and B are then alive and C dead, the probability of which in the same year is $\frac{a' b'}{a b} \times (1 - \frac{c'}{c})$: 3° that A and C are then alive and B dead, the probability of which in the same year is $\frac{a' c'}{a c} \times (1 - \frac{b'}{b})$: 4° that

* Simpson, Prob. 3. Morgan, Prob. 7. Dodson, vol. ii. Ques. 87.

† All the cases of *two* lives; and likewise the *remaining cases of three lives*, may be solved by means of the preceding problems:

B and C are then alive and A dead, the probability of which in the same year is $\frac{b'c'}{bc} \times (1 - \frac{a'}{a})$. The sum of all these chances, therefore, or $\frac{a'b'c'}{abc} + \frac{a'b'}{ab} \times (1 - \frac{c'}{c}) + \frac{a'c'}{ac} \times (1 - \frac{b'}{b}) + \frac{b'c'}{bc} \times (1 - \frac{a'}{a}) = \frac{a'b}{ab} + \frac{a'c'}{ac} + \frac{b'c'}{bc} - \frac{2a'b'c'}{abc}$, being multiplied by $(1 + \rho)^{-1}$ will give the present value of the first year's rent, or the *expectation* of receiving such sum on the contingency that any two of the original lives will outlive the first year. By a similar method of reasoning it will be found that $\frac{a''b''}{ab} + \frac{a''c''}{ac} + \frac{b''c''}{bc} - \frac{2a''b''c''}{abc}$, multiplied by $(1 + \rho)^{-2}$, will give the present value of the second year's rent, or the *expectation* of receiving such sum on the contingency that any two of the original lives will outlive the second year. Also that $\frac{a'''b'''}{ab} + \frac{a'''c'''}{ac} + \frac{b'''c'''}{bc} - \frac{2a'''b'''c'''}{abc}$, multiplied by $(1 + \rho)^{-3}$, will give the present value of the third year's rent; and so on, for all the subsequent years to the utmost extremity of human life; the sum of all which expectations, or the series

$$\begin{aligned} & (1 + \rho)^{-1} \times \left[\frac{a'b'}{ab} + \frac{a'c'}{ac} + \frac{b'c'}{bc} - \frac{2a'b'c'}{abc} \right] + \\ & (1 + \rho)^{-2} \times \left[\frac{a''b''}{ab} + \frac{a''c''}{ac} + \frac{b''c''}{bc} - \frac{2a''b''c''}{abc} \right] + \\ & (1 + \rho)^{-3} \times \left[\frac{a'''b'''}{ab} + \frac{a'''c'''}{ac} + \frac{b'''c'''}{bc} - \frac{2a'''b'''c'''}{abc} \right] + \\ & \qquad \qquad \qquad \&c \qquad \qquad \qquad \&c \end{aligned}$$

will be the total present value of the annuity required.

§ 64. But, if we take the value of each perpendicular series, as in the last problem, and substitute the characters given in Prob. I, cor. 1, the above expression will become $AB + AC + BC - 2ABC$: whence the following rule for this particular case. From the sum of the values of an annuity on each pair of joint lives, take twice the value of an annuity on the three joint lives; the difference will be the required value of an annuity to continue as long as any two of the lives are in being together.

For examples of the use and application of this problem, see Question X in Chapter XII.

§ 65. By a similar method of proceeding we might ascertain the value of an annuity granted on four lives, but to continue only as long as any *two* or *three* of them are in being together (the other cases of such lives being already solved by the two preceding problems); and, in general, the value of an annuity on any number of lives, to continue only as long as any number of them are in being together. As it rarely happens, however, that more than three lives are concerned in any practical cases, I shall not trouble the reader with the steps of the process, but shall merely state the result of the investigation in one general formula, which will comprehend in one view all the possible cases mentioned in the three problems here given*. Let the sum of the values of an annuity on

* It will be readily seen that Problem I and II are only particular cases of this third problem.

every n joint lives be denoted by $\bar{S}$; on every $(n+1)$ joint lives, by $\bar{S}^I$; on every $(n+2)$ joint lives, by $\bar{S}^{II}$; on every $(n+3)$ joint lives, by $\bar{S}^{III}$; &c.* then will the value of an annuity granted upon any number of lives, to continue only as long as any n of them are in being together, be expressed by $\bar{S} - n.\bar{S}^I + \frac{n}{1}.\frac{n+1}{2}.\bar{S}^{II} - \frac{n}{1}.\frac{n+1}{2}.\frac{n+2}{3}.\bar{S}^{III} + \&c.$

COROLLARY I.

§ 66. If the annuity is not to continue during the whole period of the given lives, but is either *Deferred* or *Temporary*, we must pursue the same method of reasoning which has been adopted in Prob. II, cor. 2 and 4: whereby it will appear that the value of a deferred annuity, to continue as long as any two out of three given lives are in being, will be expressed by $(AB)^d + (AC)^d + (BC)^d - 2(ABC)^d$. Therefore if we substitute the values of deferred annuities on the given lives (in the formula in § 64) instead of the values of annuities on the whole continuance of those lives, we shall obtain the value of the deferred annuity depending on the contingency alluded to in the problem.

* That is, on every combination which can be made of the given lives, by combining n , $(n+1)$, $(n+2)$, $(n+3)$, &c. lives together at a time.

And this value, being subtracted from the value of a similar annuity on the whole continuance of the lives, will give the value of a similar *Temporary annuity* on any two out of three lives.

COROLLARY II.

§ 67. If such deferred annuity however depends on the *joint* continuance of all the given lives to the end of the given term, the value of it will be equal to the value of a similar annuity on the same number of lives each older by the given term than such lives, multiplied by the expectation that the *joint* lives shall receive £1 at the end of that term: and this case must be carefully distinguished from that mentioned in the first part of the preceding corollary.

CHAPTER III.

On Reversions.

§ 68. A REVERSIONARY life annuity is a term applied to such periodical sums of money, depending on any given lives, as are not payable till after a given term, or till after the extinction of any other given lives. Of the former kind are all *deferred* life annuities, mentioned in Prob. I, cor. 3; but it is my intention now to treat only of the latter kind: and I would here observe, that I shall continue to designate the former by the title of *Deferred* life annuities, applying the term *Reversionary* life annuities to such life annuities only as are not to be enjoyed till after the extinction of some other life. The several cases relating to this subject may be comprised in the four following problems.

PROBLEM IV.*

§ 69. To find the value of an annuity depending on any number of joint lives A B C &c, after the extinction of any number of other joint lives P Q R &c.†

* Simpson, Prob. 5.

† The lives P, Q, R, are said to be in *possession*; in opposition to A, B, C, which are said to be in *reversion*.

SOLUTION.

Let the probability of the joint lives A B C* continuing 1, 2, 3, &c, years, be respectively denoted by $\frac{a' b' c'}{a b c}$, $\frac{a'' b'' c''}{a b c}$, $\frac{a''' b''' c'''}{a b c}$, &c, as in § 24; and let the probability of the joint lives P Q R continuing 1, 2, 3, &c, years be denoted by $\frac{p' q' r'}{p q r}$, $\frac{p'' q'' r''}{p q r}$, $\frac{p''' q''' r'''}{p q r}$, &c, respectively.

Now the chance, which the joint lives A B C have of receiving the annuity in any one year, will depend upon their living to the end of that year and on the joint lives P Q R becoming extinct before the end of that period. The probability of this event happening

in the first year is $\frac{a' b' c'}{a b c} (1 - \frac{p' q' r'}{p q r})$; which being multiplied by $(1 + e)^{-1}$ will give the present value of the first year's rent, or the expectation of receiving such sum at the end of the first year. By a similar method

of reasoning it will be found that $\frac{a'' b'' c''}{a b c} (1 - \frac{p'' q'' r''}{p q r})$,

multiplied by $(1 + e)^{-2}$ will denote the present value

of the second year's rent; and that $\frac{a''' b''' c'''}{a b c} (1 - \frac{p''' q''' r'''}{p q r})$,

* In the solution of this and the following problems, I shall confine the case to *three* lives only, in order to avoid confusion by the constant repetition of the character &c. From what has preceded, it will plainly appear that the conclusions which are drawn from this assumption will be true, whatever be the number of lives involved in the process.

multiplied by $(1+\rho)^{-3}$, will denote the present value of the third year's rent; and so on for all the subsequent years to the utmost extent of human life: the sum of all which terms, or the series

$$\begin{aligned} & \frac{1}{(1+\rho)} \left[\frac{a' b' c'}{a b c} - \frac{a' b' c' p' q' r'}{a b c p q r} \right] + \\ & \frac{1}{(1+\rho)^2} \left[\frac{a'' b'' c''}{a b c} - \frac{a'' b'' c'' p'' q'' r''}{a b c p q r} \right] + \\ & \frac{1}{(1+\rho)^3} \left[\frac{a''' b''' c'''}{a b c} - \frac{a''' b''' c''' p''' q''' r'''}{a b c p q r} \right] + \\ & \qquad \qquad \qquad \&c \qquad \qquad \&c, \end{aligned}$$

will be the total present value of the annuity required. But the sum of these two perpendicular series is evidently equal to $ABC - ABCPQR$.

§ 70. Whence it follows that if we subtract the value of an annuity on all the joint lives, from the value of an annuity on the joint lives in reversion; the remainder will be the value of the reversionary annuity required.

PROBLEM V.*

§ 71. To find the value of an annuity depending on any number of joint lives $A B C \&c$, after the extinction of the longest of any number of other lives $P, Q, R, \&c$.

* Simpson, Prob. 6.

SOLUTION.

Let the probabilities of the given lives continuing 1, 2, 3, &c, years be denoted by the same characters as in the last problem. Now the chance, which the joint lives A B C have of receiving the annuity at the end of any one year, will depend upon their living to the end of that year, and on all the lives P, Q, R, becoming extinct before the end of that period. The probability of this event happening in the first year is $\frac{a' b' c'}{a b c} (1 - \frac{p}{p}) \times (1 - \frac{q}{q}) \times (1 - \frac{r}{r})^*$, which being multiplied by $(1 + \rho)^{-1}$ will give the present value of the first year's rent. In like manner, $\frac{a'' b'' c''}{a b c} (1 - \frac{p}{p}) \times (1 - \frac{q}{q}) \times (1 - \frac{r}{r})$ multiplied by $(1 + \rho)^{-2}$, and $\frac{a''' b''' c'''}{a b c} (1 - \frac{p}{p}) \times (1 - \frac{q}{q}) \times (1 - \frac{r}{r})$ multiplied by $(1 + \rho)^{-3}$, will give the present value of the second and third year's rents respectively: and so on for all the subsequent years, to the utmost extent of human life; the sum of all which terms will be the present value of the annuity required. But these expressions, reduced to their simplest terms, are equal to the series

$$\begin{aligned} & \frac{a' b' c'}{a b c (1 + \rho)} \left[1 - \frac{p}{p} - \frac{q}{q} - \frac{r}{r} + \frac{p' q'}{p q} + \frac{p' r}{p r} + \frac{q' r}{q r} - \frac{p' q' r}{p q r} \right] + \\ & \frac{a'' b'' c''}{a b c (1 + \rho)^2} \left[1 - \frac{p}{p} - \frac{q}{q} - \frac{r}{r} + \frac{p'' q''}{p q} + \frac{p'' r}{p r} + \frac{q'' r}{q r} - \frac{p'' q'' r}{p q r} \right] + \\ & \frac{a''' b''' c'''}{a b c (1 + \rho)^3} \left[1 - \frac{p}{p} - \frac{q}{q} - \frac{r}{r} + \frac{p''' q'''}{p q} + \frac{p''' r}{p r} + \frac{q''' r}{q r} - \frac{p''' q''' r}{p q r} \right] + \\ & \qquad \qquad \qquad \text{\&c} \qquad \qquad \qquad \text{\&c} \qquad \qquad \qquad \text{\&c,} \end{aligned}$$

* See Chapter I, § 9.

the sum of which is evidently equal to $ABC - ABCP - ABCQ - ABCR + ABCPQ + ABCPR + ABCQR - ABCPQR$.

§ 72. Whence it follows that the value of a reversionary annuity on any number of joint lives ABC &c, after the extinction of the longest of any number of other lives P, Q, R, &c, is equal to the value of an annuity on all the joint lives in reversion; *minus* the sum of the values of an annuity on all the joint lives, arising from the combination of all the joint lives in reversion with each ONE of the other lives; *plus* the sum of the values of an annuity on all the joint lives, arising from the combination of all the joint lives in reversion with each TWO of the other lives; *minus* the sum of the values of an annuity on all the joint lives, arising from the combination of all the joint lives in reversion with each THREE of the other lives; and so on.

PROBLEM VI.*

§ 73. To find the value of an annuity on the longest of any number of lives A, B, C, &c, after the extinction of any number of joint lives P Q R &c.

SOLUTION.

Let the probabilities of the given lives continuing 1, 2, 3, &c, years be denoted by the same characters

* Simpson, Prob. 7.

as in the preceding problems. Now, the chance which any one of the lives A, B, C, has of receiving the annuity at the end of any one year, will depend upon either of them living to the end of that year, and on the extinction of the joint lives P Q R before the end of that period. The probability of this happening in the first year is $\left[1 - \left(1 - \frac{a'}{a}\right) \times \left(1 - \frac{b'}{b}\right) \times \left(1 - \frac{c'}{c}\right)\right] \times \left(1 - \frac{p'q'r'}{pqr}\right)$, which being multiplied by $(1 + \rho)^{-1}$ will give the present value of the first year's rent. In like manner it will be found that $\left[1 - \left(1 - \frac{a''}{a}\right) \times \left(1 - \frac{b''}{b}\right) \times \left(1 - \frac{c''}{c}\right)\right] \times \left(1 - \frac{p''q''r''}{pqr}\right)$, multiplied by $(1 + \rho)^{-2}$, and that $\left[1 - \left(1 - \frac{a'''}{a}\right) \times \left(1 - \frac{b'''}{b}\right) \times \left(1 - \frac{c'''}{c}\right)\right] \times \left(1 - \frac{p'''q'''r'''}{pqr}\right)$ multiplied by $(1 + \rho)^{-3}$, will give the present value of the second and third year's rents respectively; and so on for all the subsequent years, to the utmost extent of human life: the sum of all which terms will be the total present value of the annuity required. But these annual expectations being reduced to their simplest terms, and arranged under each other as in the preceding problems, will form fourteen collateral series, the sum of all which will be found equal to $A + B + C - AB - AC - BC + ABC - PQRA - PQRB - PQRC + PQRAB + PQRAC + PQRBC - PQRABC$.

§ 74. Whence it follows that the value of a reversionary annuity, on the longest of any number of lives after any number of joint lives, is equal to the value of an annuity on the longest of all the lives in reversion; *minus* the sum of the values of an annuity on all the joint lives arising from the combination of all the joint lives in possession with each ONE of the other lives; *plus* the sum of the values of an annuity on all the joint lives arising from the combination of all the joint lives in possession with each TWO of the other lives; *minus* the sum of the values of an annuity on all the joint lives arising from the combination of all the joint lives in possession with each THREE of the other lives; and so on.

PROBLEM VII.*

§ 75. To find the value of an annuity on the longest of any number of lives A, B, C, &c, after the extinction of the longest of any number of other lives P, Q, R, &c.

SOLUTION.

From what has been already said in the preceding Problems, it will be evident that the annuity, here alluded to, is to continue during the longest of all the given lives A, B, C, P, Q, R; and such would be the

* Simpson, Prob. 4.

value of it, were the lives A, B, C , to come into possession immediately. But as they are to receive nothing during the existence of any one of the lives P, Q, R ; the value of an annuity on the longest of their lives must consequently be subtracted.

Whence it follows that, if we subtract the value of an annuity on the longest of the lives in possession from the value of an annuity on the longest of all the lives concerned, the remainder will be the value of the reversionary annuity required.

SCHOLIUM.

§ 76. By the help of these four problems may all the various cases in reversionary annuities be solved. They have been stated at length, in order to give a general view of the subject: but it will readily appear that the combinations of lives, thence arising, are much more numerous than occur in any practical cases; and a ready solution may therefore not immediately present itself amidst the multiplicity of symbols. It rarely happens that more than three lives are involved in any questions of this kind: and in order to prevent any difficulty or confusion in referring to the problems for a solution of such questions, I have selected all the possible cases in which not more than three lives are concerned; to which I have added the algebraic solution of the same: where A, B, P, Q , denote the same as in the preceding problems. These cases are

five in number:* viz. to find the value of an annuity

1° On a single life A after another single life P: in which case the value of the reversionary annuity is equal to $A - AP$.

2° On a single life A after the longest of two lives P, Q, in which case the value of the reversionary annuity is equal to $A - AP - AQ + APQ$.

3° On the longest of two lives A, B, after a single life P: in which case the value of the reversionary annuity is equal to $A + B - AB - AP - BP + ABP$.

4° On a single life A after two joint lives PQ: in which case the value of the reversionary annuity is equal to $A - APQ$.

5° On two joint lives AB after a single life P: in which case the value of the reversionary annuity is equal to $AB - ABP$.

For examples of these several cases, see Questions XIII to XVII in Chapter XII.

COROLLARY I.

§ 77. If the contingency of receiving the annuity is not to continue during the whole period of the given lives, but is either *Deferred*† or *Temporary*, we must revert to what has been said in Prob. I, cor. 3 and 4, and substitute the values of such deferred and tem-

* Simpson, Prob. 13 to 17; and his *Sup.* Prob. 15 to 19; Dodson, vol. ii. Ques. 94, 95, 97, 99, and 101; and vol. iii. Ques. 20, 27, 28 and 29. Also De Moivre, Prob. 7 and 8, for the first and second cases.

† Price, Note (D).

porary annuities, deduced from the rules there given, instead of the values of annuities on the whole continuance of those lives: whence by proceeding with these substituted values according to the rules above stated, we shall obtain the required value of the reversionary annuity accordingly.

But this rule may be rendered more convenient for practice by adopting the principles laid down in Prob. II, cor. 2 and 4. Thus, in the first case given in the Scholium, if the contingency is *deferred* for n years, the value of the reversionary annuity will be $(A)^d - (AP)^d = A \times \frac{a}{a} (1 + e)^{-n} - A P \times \frac{a\pi}{ap} (1 + e)^{-n}$.^{*} And this value subtracted from $A - AP$ will give the value of a similar *temporary* reversionary annuity. The same method of solution will apply to the remaining cases given in the Scholium.

For examples of the use and application of this corollary, see Questions XVIII and XIX in Chapter XII.

COROLLARY II.†

§ 78. If a reversionary annuity, the contingency of enjoying which is deferred for any number of years, depends on the *joint* continuance of all the lives to

* From the method of notation adopted throughout the present work it will readily appear that π denotes in this case the number of persons living at an age n years older than P .

† Price, Note (C).

the end of that term ; the value of it will be equal to the value of a reversionary annuity depending on the same number of lives each older by the given term than the given lives, multiplied by the probability that *all* the joint lives shall continue so long, and also by the present value of £1 due at the end of that term. And this case should be carefully distinguished from the deferred reversionary annuity mentioned in the preceding corollary,

For examples of the use and application of this corollary, see the Scholium to Question XIX in Chapter XII.

CHAPTER IV.

On Survivorships.

§ 79. IN the preceding chapters I have considered the value of annuities as depending on the continuance of any number of lives, out of any number of given lives; and also the value of reversionary annuities on any number of lives, after the extinction of any number of other lives. I come now to questions of a mixed nature, where the value of the annuity not only depends on the continuance of the given lives, but also on any survivorship between them. In cases of this kind the annuity is frequently enjoyed in different proportions by the persons on whose lives the same is granted; and therefore they are capable of great variety. The nature and extent of such questions will best appear from the following problems.

PROBLEM VIII.*

§ 80. A, B and C agree amongst themselves to purchase an annuity on the longest of their lives,

* Simpson, Prob. 20; also his *Sup.* Prob. 23. De Moivre, Prob. 9. Dodson, vol. iii. Ques. 73.

which is to be equally divided between them whilst they are all living; but on the decease of either of them it is to be equally divided between the two survivors, during their joint lives; and then to belong entirely to the last survivor for his life: To find the value of their respective shares, or the proportion which each person ought to contribute towards the purchase.

SOLUTION.

Let the probabilities of the given lives continuing 1, 2, 3, &c, years be severally denoted by the same characters as in § 23: and let us first determine the share of A. Now the expectation of A, on what he may happen to receive at the end of any one year, may be considered in four parts, as depending on so many different events: 1° A, B and C may be all living, the probability of which at the end of the first year is $\frac{a' b' c'}{a b c}$, in which case he will receive $\frac{1}{3}$ of the annuity, or $\frac{1}{3} (1 + \rho)^{-1}$; therefore $(1 + \rho)^{-1} \times \frac{a' b' c'}{3 a b c}$ will be the value of this expectation: 2° A and B may be living and C dead, the probability of which at the end of the first year is $\frac{a' b'}{a b} \times (1 - \frac{c'}{c})$, in which case he will receive $\frac{1}{2}$ the annuity, or $\frac{1}{2} (1 + \rho)^{-1}$; therefore $(1 + \rho)^{-1} \times \frac{a' b'}{2 a b} \times (1 - \frac{c'}{c})$ will be the value of this expectation: 3° A and C may be living and

B dead, the probability of which at the end of the first year is $\frac{a'c}{ac} \times (1 - \frac{b'}{b})$, in which case he will likewise receive $\frac{1}{2}$ the annuity, or $\frac{1}{2}(1+\rho)^{-1}$; therefore $(1+\rho)^{-1} \times \frac{a'c}{2ac} \times (1 - \frac{b'}{b})$ will be the value of this expectation: 4° A may be the only person living, the probability of which at the end of the first year is $\frac{a'}{a} \times (1 - \frac{b'}{b}) \times (1 - \frac{c'}{c})$, in which case he comes in for the whole annuity, therefore $(1+\rho)^{-1} \times \frac{a'}{a} \times (1 - \frac{b'}{b}) \times (1 - \frac{c'}{c})$ will be the value of this expectation. And the sum of all these values, or $(1+\rho)^{-1} \times (\frac{a'}{a} - \frac{a'b'}{2ab} - \frac{a'c'}{2ac} + \frac{a'b'c'}{3abc})$ will be the total value of the expectation of A, on what he may happen to receive at the end of the first year.

By pursuing the same steps it will be found that $(1+\rho)^{-2} \times (\frac{a''}{a} - \frac{a''b''}{2ab} - \frac{a''c''}{2ac} + \frac{a''b''c''}{3abc})$ will denote the value of his expectation on what he may happen to receive at the end of the second year; and that $(1+\rho)^{-3} \times (\frac{a'''}{a} - \frac{a'''b'''}{2ab} - \frac{a'''c'''}{2ac} + \frac{a'''b'''c'''}{3abc})$ will denote the value of his expectation on what he may happen to receive at the end of the third year: and so on to the utmost extent of human life. The sum of all which terms, or the series

$$\begin{aligned}
 & (1+\rho)^{-1} \times \left[\frac{a'}{a} - \frac{a'b'}{2ab} - \frac{a'c'}{2ac} + \frac{a'b'c'}{3abc} \right] + \\
 & (1+\rho)^{-2} \times \left[\frac{a''}{a} - \frac{a''b''}{2ab} - \frac{a''c''}{2ac} + \frac{a''b''c''}{3abc} \right] + \\
 & (1+\rho)^{-3} \times \left[\frac{a'''}{a} - \frac{a'''b'''}{2ab} - \frac{a'''c'''}{2ac} + \frac{a'''b'''c'''}{3abc} \right] + \\
 & \qquad \qquad \qquad \&c \qquad \qquad \qquad \&c
 \end{aligned}$$

will be the total value of his interest in the annuity, or the share which he ought to contribute towards the purchase. But the sum of these several perpendicular series is equal to $A - \frac{1}{2}AB - \frac{1}{2}AC + \frac{1}{3}ABC$.

§ 81. As to the share of B, or C, it is evident that their expectation of receiving the annuity in any one year, will depend on the same events, *mutatis mutandis*, as that of A: whence it follows that by substituting the values, thence arising, in the general expression above given, we shall have $B - \frac{1}{2}AB - \frac{1}{2}BC + \frac{1}{3}ABC$ for the value of B's share; and $C - \frac{1}{2}AC - \frac{1}{2}BC + \frac{1}{3}ABC$ for the value of C's share in the given annuity: whence the following rule for either case.

§ 82. *From the value of an annuity on the given life subtract half the sum of the values of an annuity on both the joint lives arising from the combination of the given life with each one of the other; and to the remainder add one third of the value of an annuity on the three joint lives: the sum will be the answer required.*

§ 83. *Example.* Suppose the three lives to be aged 20, 30, and 40; the rate of interest to be 4 per cent, and the probabilities of living as at *Northampton*. Then by referring to the Tables at the end of this work it will be found that the value of the share of each person in this annuity will be as under: viz. that of

$$A = 16.033 - \frac{1}{2}(11.873 + 10.924) + \frac{1}{3} \times 8.986 = 7.630$$

$$B = 14.781 - \frac{1}{2}(11.873 + 10.490) + \frac{1}{3} \times 8.986 = 6.595$$

$$C = 13.197 - \frac{1}{2}(10.924 + 10.490) + \frac{1}{3} \times 8.986 = 5.485$$

and the sum of all these respective shares, or 19.710 is the value of the annuity on the longest of the three lives; or the total value of the purchase.

COROLLARY I.*

§ 84. If only two lives, A and B, are concerned in the purchase, (during whose joint continuance the annuity is to be enjoyed equally between them, but on the decease of either of them it is to belong wholly to the survivor) the value of the share of A will be $A - \frac{1}{2}AB$; and that of B will be $B - \frac{1}{2}AB$: whence the following rule for two lives.

§ 85. *From the value of an annuity on the given life, subtract half the value of an annuity on the two joint lives: the remainder will be the share of the given life required.*

* Simpson, Prob. 18, and his *Sup. Prob.* 21. Dodson, vol. iii. Ques. 72.

For examples of the use and application of this corollary, see Question XX in Chapter XII.

COROLLARY II.

§ 86. On the other hand, let the number of lives concerned in the purchase be ever so great, the share of any given life may be readily determined, provided the annuity be always equally divided among the surviving lives. For, let G denote the value of an annuity on the given life; G^I the sum of the values of an annuity on all the joint lives arising from combining the given life with each *one* of the others; G^{II} the sum of the values of an annuity on all the joint lives arising from combining the given life with each *two* of the others; G^{III} the sum of the values of an annuity on all the joint lives arising from combining the given life with each *three* of the others; and so on. Then will $G - \frac{1}{2}G^I + \frac{1}{3}G^{II} - \frac{1}{4}G^{III} + \&c$ denote the share of the given life, or the value which he ought to contribute towards the purchase.

PROBLEM IX.*

§ 87. A, B and C agree to purchase an annuity on the longest of their lives, which is to be divided amongst them in the following manner: A and B are to enjoy it equally during their joint lives; but on the

* Dodson, vol. iii. Ques. 75.

decease of either of them it is to be equally divided between A and C, or B and C, the two remaining persons; and lastly, to be enjoyed wholly by the survivor: To find the value of their respective shares.

SOLUTION.

The expectation of A on what he may happen to receive at the end of any one year, may be considered in three parts, as depending on so many different events: 1° A and B may be both alive, the probability of which at the end of the first year is $\frac{a'b'}{ab}$, in which case he will receive $\frac{1}{2}$ the annuity: 2° A and C may be living and B dead, the probability of which at the end of the first year is $\frac{a'c'}{ac} \times (1 - \frac{b'}{b})$ in which case he will also receive $\frac{1}{2}$ the annuity: 3° A may be the only person living, the probability of which at the end of the first year is $\frac{a'}{a} \times (1 - \frac{b'}{b}) \times (1 - \frac{c'}{c})$, in which case he will come in for the whole annuity. Now, the sum of all these values, being multiplied by $(1+p)^{-1}$ will give $(1+p)^{-1} \times (\frac{a'}{a} - \frac{a'b'}{2ab} - \frac{a'c'}{2ac} + \frac{a'b'c'}{2abc})$ for the total value of the expectation of A on what he may happen to receive at the end of the first year.

By a similar process we may find the value of his expectation on what he may happen to receive at the end of the second, third, and every subsequent year,

to the utmost extent of human life. The sum of all which terms, or the series

$$\begin{aligned}
 & (1+\rho)^{-1} \times \left[\frac{a}{a} - \frac{a'b}{2ab} - \frac{a'c}{2ac} + \frac{a'b'c}{2abc} \right] + \\
 & (1+\rho)^{-2} \times \left[\frac{a''}{a} - \frac{a''b}{2ab} - \frac{a''c}{2ac} + \frac{a''b''c}{2abc} \right] + \\
 & (1+\rho)^{-3} \times \left[\frac{a'''}{a} - \frac{a'''b}{2ab} - \frac{a'''c}{2ac} + \frac{a'''b'''c}{2abc} \right] + \\
 & \qquad \qquad \qquad \&c \qquad \qquad \qquad \&c,
 \end{aligned}$$

will denote the total value of his interest in the annuity. But, the sum of these several perpendicular series is evidently equal to $A - \frac{1}{2}AB - \frac{1}{2}AC + \frac{1}{2}ABC$: whence we have the following rule.

§ 88. *From the value of an annuity on the life A, subtract half the sum of the values of an annuity on the joint lives AB, and AC; to the remainder add half the value of an annuity on the three joint lives ABC: the sum will be the share which A ought to contribute towards the purchase.*

§ 89. With respect to the value of B's interest, it is evident that his expectation of receiving the annuity in any one year will depend on the same events, *mutatis mutandis*, as that of A: wherefore by substituting the values, thence arising, in the general expression above given, we shall have $B - \frac{1}{2}AB - \frac{1}{2}BC + \frac{1}{2}ABC$ for the value of B's interest: whence the following rule.

§ 90. *From the value of an annuity on the life B subtract half the sum of the values of an annuity on the joint lives AB, and BC; to the remainder add half the value of an annuity on the three joint lives ABC; the sum will be the share which B ought to contribute towards the purchase.*

§ 91. But with respect to C's interest, it will appear that his expectation of receiving the annuity in any one year may be considered in the three following points, as depending on so many different events: 1° A and C may be living and B dead, the probability of which at the end of the first year is $\frac{a'c'}{ac} \times (1 - \frac{b}{b})$, in which case he will receive $\frac{1}{2}$ the annuity: 2° B and C may be living and A dead, the probability of which at the end of the first year is $\frac{b'c'}{bc} \times (1 - \frac{a}{a})$, in which case he will receive also $\frac{1}{2}$ the annuity: 3° C may be the only person living, the probability of which at the end of the first year is $\frac{c'}{c} \times (1 - \frac{a}{a}) \times (1 - \frac{b}{b})$, in which case he will receive the whole annuity. The sum of these values therefore, being multiplied by $(1 + \rho)^{-1}$, will give $(1 + \rho)^{-1} \times (\frac{c'}{c} - \frac{a'c'}{2ac} - \frac{b'c'}{2bc})$ for the total value of the expectation of C on what he may happen to receive at the end of the first year.

By pursuing the same steps, we may find the value

of his expectation on what he may happen to receive at the end of the second, third and every subsequent year to the utmost extent of human life. The sum of all which terms will be the total value of his interest in the annuity; and will be evidently equal to $C - \frac{1}{2}AC - \frac{1}{2}BC$: whence the following rule.

§ 92. *From the value of an annuity on the life C, subtract half the sum of the values of an annuity on the joint lives AC, and BC; the remainder will be the share which C ought to contribute towards the purchase.*

§ 93. *Example.* Let the three lives be aged 20, 30, and 40; the rate of interest 4 per cent, and the probabilities of living as at *Northampton*. The value of the share of each person in this annuity will be as under: viz. that of

$$A = 16.033 - \frac{1}{2}(11.873 + 10.924) + \frac{1}{2} \times 8.986 = 9.127$$

$$B = 14.781 - \frac{1}{2}(11.873 + 10.490) + \frac{1}{2} \times 8.986 = 8.093$$

$$C = 13.197 - \frac{1}{2}(10.924 + 10.490) \dots \dots = 2.490$$

And the sum of all these values is the value of an annuity on the longest of the three lives.

PROBLEM X.*

§ 94. A, B and C agree to purchase an annuity

* Simpson, Prob. 19. Dodson, vol. iii. Ques. 74.

on the longest of their lives, to be divided amongst them in the following manner: A and B are to enjoy it equally during their joint lives; if A dies first, then B and C are to enjoy it equally during their joint lives, and the survivor of them is to have the whole; but if B dies first, then A is to enjoy the whole during his life, and after his decease it is to devolve wholly to C: To find the value of their respective shares.

SOLUTION.

The expectation of A, on what he may happen to receive at the end of any one year, may be considered in two points, as depending on so many different events; 1° A and B may be both alive, the probability of which at the end of the first year is $\frac{a'b'}{ab}$, in which case he will receive $\frac{1}{2}$ the annuity: 2° A may be living and B dead, the probability of which at the end of the first year is $\frac{a'}{a} \times (1 - \frac{b'}{b})$, in which case he will enjoy the whole annuity. The sum of these two values being multiplied by $(1 + \rho)^{-1}$ will give $(1 + \rho)^{-1} \times (\frac{a'}{a} - \frac{a'b'}{2ab})$ for the total value of the expectation of A on what he may happen to receive at the end of the first year.

By a similar process we may find the value of his expectation on what he may happen to receive at the end of the second, third, and every subsequent year—

to the utmost extent of human life. The sum of all which terms will be the total value of his interest in the annuity; and will be evidently equal to $A - \frac{1}{2}AB$: whence we have the following rule.

§ 95. *From the value of an annuity on the life A, subtract half the value of an annuity on the joint lives AB: the remainder will be the value which A ought to contribute towards the purchase.*

§ 96. In order to determine the share of B, it should be observed that his expectation on what he may happen to receive at the end of any one year may be considered in three points, as depending on so many different events: 1° A and B may be both alive, the probability of which at the end of the first year is $\frac{ab'}{ab}$, in which case he will receive $\frac{1}{2}$ the annuity: 2° B and C may be alive and A dead, the probability of which at the end of the first year is $\frac{bc'}{bc} \times (1 - \frac{a'}{a})$, in which case also he will receive $\frac{1}{2}$ the annuity: 3° B may be the only person living, the probability of which at the end of the first year is $\frac{b}{b} \times (1 - \frac{a'}{a}) \times (1 - \frac{c'}{c})$, in which case he will enjoy the whole annuity. And the sum of all these values, being multiplied by $(1+p)^{-1}$, will give $(1+p)^{-1}(\frac{b}{b} - \frac{a'b'}{2ab} -$

on the longest of their lives, to be divided amongst them in the following manner: A and B are to enjoy it equally during their joint lives; if A dies first, then B and C are to enjoy it equally during their joint lives, and the survivor of them is to have the whole; but if B dies first, then A is to enjoy the whole during his life, and after his decease it is to devolve wholly to C: To find the value of their respective shares.

SOLUTION.

The expectation of A, on what he may happen to receive at the end of any one year, may be considered in two points, as depending on so many different events; 1° A and B may be both alive, the probability of which at the end of the first year is $\frac{a'b'}{ab}$, in which case he will receive $\frac{1}{2}$ the annuity: 2° A may be living and B dead, the probability of which at the end of the first year is $\frac{a'}{a} \times (1 - \frac{b'}{b})$, in which case he will enjoy the whole annuity. The sum of these two values being multiplied by $(1 + \rho)^{-1}$ will give $(1 + \rho)^{-1} \times (\frac{a'}{a} - \frac{a'b'}{2ab})$ for the total value of the expectation of A on what he may happen to receive at the end of the first year.

By a similar process we may find the value of his expectation on what he may happen to receive at the end of the second, third, and every subsequent year,

$(1 - \frac{b}{b})$, in which case he will enjoy the whole annuity. And the sum of these two values, being multiplied by $(1+p)^{-1}$, will give $(1+p)^{-1} \times (\frac{c}{c} - \frac{ac}{ac} - \frac{bc}{2bc} + \frac{abc}{2abc})$ for the total value of the expectation of C, on what he may happen to receive at the end of the first year.

By a similar method of proceeding we may find the value of his expectation for the second, third, and every subsequent year to the utmost extent of human life. The sum of all which terms will express the total value of his interest in the annuity; and will be evidently equal to $C - AC - \frac{1}{2}BC + \frac{1}{2}ABC$: whence the following rule.

§ 99. *From the value of an annuity on the life C, subtract the value of an annuity on the joint lives AC, and also half the value of an annuity on the joint lives BC; to the remainder add half the value of an annuity on the joint lives ABC: the sum will be the share which C ought to contribute towards the purchase.*

§ 100. *Example.* Suppose the three lives to be aged 20, 30, and 40; the rate of interest 4 per cent, and the probabilities of living as at *Northampton*. The value of the share of each person in this annuity will be as under: viz. that of

$\frac{b'c'}{2bc} + \frac{a'b'c'}{2abc}$) for the total value of the expectation of B, on what he may happen to receive at the end of the first year.

By a similar process we may find the value of his expectation for the second, third and every subsequent year, to the utmost extent of human life. The sum of all which terms will be the total value of his interest in the annuity; and will be evidently equal to $B - \frac{1}{2}AB - \frac{1}{2}BC + \frac{1}{2}ABC$: whence the following rule.

§ 97. *From the value of an annuity on the life B, subtract half the sum of the values of an annuity on the joint lives AB, and BC; to the remainder add half the value of an annuity on the three joint lives ABC: the sum will be the value which B ought to contribute towards the purchase.**

§ 98. It remains now to determine the value of the share of C, whose expectation of receiving the annuity in any one year may be considered in two points: 1° B and C may be both alive and A dead, the probability of which at the end of the first year is $\frac{b'c'}{bc} \times (1 - \frac{a'}{a})$, in which case he will receive $\frac{1}{2}$ the annuity: 2° C only may be living, the probability of which at the end of the first year is $\frac{c'}{c} \times (1 - \frac{a'}{a}) \times$

* The value of B's interest in this annuity is the same as that in the preceding Problem.

$(1 - \frac{b}{b})$, in which case he will enjoy the whole annuity. And the sum of these two values, being multiplied by $(1 + e)^{-1}$, will give $(1 + e)^{-1} \times (\frac{c}{c} - \frac{ac}{ac} - \frac{bc}{2bc} + \frac{abc}{2abc})$ for the total value of the expectation of C, on what he may happen to receive at the end of the first year.

By a similar method of proceeding we may find the value of his expectation for the second, third, and every subsequent year to the utmost extent of human life. The sum of all which terms will express the total value of his interest in the annuity; and will be evidently equal to $C - AC - \frac{1}{2}BC + \frac{1}{2}ABC$: whence the following rule.

§ 99. *From the value of an annuity on the life C, subtract the value of an annuity on the joint lives AC, and also half the value of an annuity on the joint lives BC; to the remainder add half the value of an annuity on the joint lives ABC: the sum will be the share which C ought to contribute towards the purchase.*

§ 100. *Example.* Suppose the three lives to be aged 20, 30, and 40; the rate of interest 4 per cent, and the probabilities of living as at *Northampton*. The value of the share of each person in this annuity will be as under: viz. that of

$$A = 16.033 - \frac{1}{2} \times 11.873 \dots\dots\dots = 10.096$$

$$B = 14.781 - \frac{1}{2}(11.873 + 10.490) + \frac{1}{2} \times 8.986 = 8.053$$

$$C = 13.197 - 10.924 - \frac{1}{2} \times 10.490 + \frac{1}{2} \times 8.986 = 1.521$$

and the sum of these three values is the value of an annuity on the longest of the three lives.

PROBLEM XI.*

§ 101. A, B and C agree to purchase an annuity on the longest of their lives, to be divided amongst them in the following manner: A is to enjoy the whole annuity during his life; but after his decease it is to be divided equally between B and C during their joint lives; and the survivor of them is to have the whole: To find the value of their respective shares.

SOLUTION.

The value of A's interest in this annuity, or the share which he ought to contribute towards the purchase, is evidently equal to the value of an annuity on his life: that is, equal to *A*.

§ 102. As to the share of B, the expectation on what he may happen to receive at the end of any one year may be considered in two points, as depending on so many different events: 1° B and C may be alive and A dead, the probability of which at the end

* Simpson's *Sup. Prob.* 22. Dodson, vol. iii. *Ques.* 76.

of the first year is $\frac{b'c'}{bc} \times (1 - \frac{a'}{a})$, in which case he will receive $\frac{1}{2}$ the annuity: 2° B may be the only person living, the probability of which at the end of the first year is $\frac{b'}{b} \times (1 - \frac{a'}{a}) \times (1 - \frac{c'}{c})$, in which case he will enjoy the whole annuity. And the sum of these two values, being multiplied by $(1 + \rho)^{-1}$, will give $(1 + \rho)^{-1} \times (\frac{b'}{b} - \frac{a'b'}{ab} - \frac{b'c'}{2bc} + \frac{a'b'c'}{2abc})$ for the total value of the expectation of B on what he may happen to receive at the end of the first year.

By a similar method of proceeding we may find the value of his expectation for the second, third and every subsequent year, to the utmost extent of human life. The sum of all which terms will be the total value of his interest in the annuity; and will be found equal to $B - AB - \frac{1}{2}BC + \frac{1}{2}ABC$: whence the following rule.

§ 103. *From the value of an annuity on the life B, subtract the value of an annuity on the joint lives AB, and also half the value of an annuity on the joint lives BC; to the remainder add half the value of an annuity on the joint lives ABC: the sum will be the value which B ought to contribute towards the purchase.*

§ 104. With respect to the share of C, it is evi-

dent that his expectation of receiving the annuity at the end of any one year will depend on the same events, *mutatis mutandis*, as that of B. Therefore, by pursuing the same steps, we shall find that the value of his interest in the annuity will be equal to $C - AC - \frac{1}{2}BC + \frac{1}{4}ABC$; whence we have the following rule.

§ 105. *From the value of an annuity on the life C, subtract the value of an annuity on the joint lives AC, and also half the value of an annuity on the joint lives BC; to the remainder add half the value of an annuity on the joint lives ABC: the sum will be the value which C ought to contribute towards the purchase.**

§ 106. *Example.* Let the three lives be aged 20, 30, and 40; the rate of interest 4 per cent, and the probabilities of living as at *Northampton*. The value of the share of each person in this annuity will be as under: viz. that of

$$A = 16.033 \dots \dots \dots = 16.033$$

$$B = 14.781 - 11.873 - \frac{1}{2} \times 10.490 + \frac{1}{2} \times 8.986 = 2.156$$

$$C = 13.197 - 10.924 - \frac{1}{2} \times 10.490 + \frac{1}{2} \times 8.986 = 1.521$$

and the sum of these three values is the value of an annuity on the longest of the three lives.

* The value of C's interest in this annuity is the same as that in the preceding problem.

PROBLEM XII.

§ 107. A, B and C agree to purchase an annuity on the longest of their lives, to be enjoyed wholly by each of them in succession: that is, A is to enjoy it first, for his life; at his decease, B is to enter upon it next; and C, last: To find the value of their respective shares.

SOLUTION.

The value of A's interest in this annuity, or the share which he ought to contribute towards the purchase, is evidently equal to the value of an annuity on his life: that is, equal to A .

§ 108. The value of B's interest, or the share which he ought to contribute towards the purchase, is equal to the value of a reversionary annuity on his life after the decease of A: that is, (by the Scholium in page 70) equal to $B - AB$.

§ 109. The value of C's interest, or the share which he ought to contribute towards the purchase, is equal to the value of a reversionary annuity on his life after the extinction of the longest of the two lives A and B: that is, (by the same Scholium) equal to $C - AC - BC + ABC$.

§ 110. *Example.* Suppose the three lives to be aged 20, 30, and 40; the rate of interest 4 per cent,

and the probabilities of living as at *Northampton*. The value of the share of each person in this annuity will be as under : viz. that of

$$A = 16.033 \dots \dots \dots = 16.033$$

$$B = 14.781 - 11.873 \dots \dots \dots = 2.908$$

$$C = 13.197 - (10.924 + 10.490) + 8.986 = .769$$

and the sum of these three values will be the value of an annuity on the longest of the three lives.

COROLLARY.

§ 111. If only two lives, A and B, are concerned in the purchase, the value of their respective shares will be the same as above given.

PROBLEM XIII.

§ 112. A, B and C agree to purchase an annuity, to continue as long as any two of their lives are in being; and which is to be equally divided between them whilst they are all living; but on the decease of either of them it is to be equally divided between the two survivors during their joint lives: To find the value of their respective shares.

SOLUTION.

The expectation of A, on what he may happen to receive at the end of any one year, may be considered in three parts, as depending on so many different events: 1° A, B and C may be all living, in which

case he will receive $\frac{1}{3}$ of the annuity: 2° A and B may be living and C dead, in which case he will receive $\frac{1}{3}$ the annuity: 3° A and C may be living and B dead, in which case he will receive also $\frac{1}{3}$ the annuity. Therefore the sum of these expectations for the first, second, third, &c, years, continued to the utmost extent of human life, will be the total value of A's share in the annuity. But the value of these several expectations has been already found in the solution to Prob. VIII; they being precisely the same as the first three which are there mentioned: and the sum of which, for every year of human life, will be found equal to $\frac{1}{2}AB + \frac{1}{2}AC - \frac{2}{3}ABC$.

§ 113. With respect to the share of B or C, their expectation of receiving the annuity in any one year will depend on the same events, *mutatis mutandis*, as that of A: whence it follows that by substituting the values, thence arising, in the general expression above given, we shall have $\frac{1}{2}AB + \frac{1}{2}BC - \frac{2}{3}ABC$, for the value of B's interest; and $\frac{1}{2}AC + \frac{1}{2}BC - \frac{2}{3}ABC$, for the value of C's interest in the given annuity: whence the following rule for determining the value of the share of any given life in the annuity.

§ 114. *Subtract two thirds of the value of an annuity on the three joint lives, from half the sum of the values of an annuity on both the joint lives arising from the combination of the given life with*

each one of the others: the remainder will be the value of the share of such given life.

§ 115. *Example.* Suppose the three lives to be aged 20, 30, and 40; interest 4 per cent, and the probabilities of living as at *Northampton*. The value of the share of each person will be as under: viz. that of

$$A = \frac{1}{2}(11.873 + 10.924) - \frac{2}{3} \times 8.986 = 5.408$$

$$B = \frac{1}{2}(11.873 + 10.490) - \frac{2}{3} \times 8.986 = 5.191$$

$$C = \frac{1}{2}(10.924 + 10.490) - \frac{2}{3} \times 8.986 = 4.716$$

and the sum of these three values, or 15.315, is the value of an annuity on the three lives, to continue as long as any two of them are in being together.

PROBLEM XIV.

§ 116. A, B and C agree to purchase an annuity, to continue as long as any two of their lives are in being; and which is to be divided amongst them in the following manner: A and B are to enjoy it equally during their joint lives; but on the death of either of them, it is to be equally divided between the two survivors, for their joint lives: To find their respective shares.

SOLUTION.

The expectation of A on what he may happen to

receive at the end of any one year may be considered in two parts, as depending on so many different events: 1° A and B may be both alive, in which case he will receive $\frac{1}{2}$ the annuity: 2° A and C may be living and B dead, in which case he will also receive $\frac{1}{2}$ the annuity. Therefore the sum of these expectations for the first, second, third, &c, years, to the utmost extent of human life will be the total value of A's share in the annuity. But the value of these several expectations has been already found in the solution to Prob. IX; they being precisely the same as the first two there mentioned: and the sum of which, for every year of human life, will be found equal to $\frac{1}{2}AB + \frac{1}{2}AC - \frac{1}{2}ABC$: whence the following rule.

§ 117. *Subtract the value of an annuity on the joint lives ABC, from the sum of the values of an annuity on each of the joint lives AB and AC: half the remainder will be the value which A ought to contribute towards the purchase.*

§ 118. With respect to the share of B, it is evident that his expectation of receiving the annuity at the end of any one year will depend on the same events, *mutatis mutandis*, as that of A: wherefore by substituting the values, thence arising, in the general expression above given, we shall have $\frac{1}{2}AB + \frac{1}{2}BC - \frac{1}{2}ABC$ for the value of B's interest in the annuity: whence the following rule.

§ 119. *Subtract the value of an annuity on the joint lives ABC, from the sum of the values of an annuity on each of the joint lives AB and BC: half the remainder will be the value which B ought to contribute towards the purchase.*

§ 120. But with respect to the share of C, it will appear that his expectation of receiving the annuity at the end of any one year may be considered under the two following points, as depending on so many different events: 1° A and C may be living and B dead, in which case he will receive $\frac{1}{2}$ the annuity: 2° B and C may be living and A dead, in which case also he will receive $\frac{1}{2}$ the annuity. Therefore the sum of these two expectations, for the first, second, third, &c, years to the utmost extent of human life, will be the total value of C's interest in the annuity. But, the value of these expectations has been already found in the solution to Prob. IX; they being precisely the same as the first two which are mentioned in the investigation of C's share in that annuity: and the sum of which, for every year of human life, will be found equal to $\frac{1}{2}AC + \frac{1}{2}BC - ABC$: whence the following rule.

§ 121. *Subtract the value of an annuity on the joint lives ABC, from half the sum of the values of an annuity on each of the joint lives AC and BC: the difference will be the share which C ought to contribute towards the purchase.*

§ 122. *Example.* Suppose the three lives to be aged 20, 30, and 40; the rate of interest 4 per cent, and the probabilities of living as at *Northampton*. The value of the share of each person in this annuity will be as under: viz. that of

$$A = \frac{1}{2}(11.873 + 10.924 - 8.986) = 6.905$$

$$B = \frac{1}{2}(11.873 + 10.490 - 8.986) = 6.689$$

$$C = \frac{1}{2}(10.924 + 10.490) - 8.986 = 1.721$$

and the sum of these three values is the total value of the annuity on the three lives, to continue as long as any two of them are in being together.

PROBLEM XV.

§ 123. A, B and C agree to purchase an annuity to continue as long as any two of them are in being together; and which is to be divided between them in the following manner: A and B are to enjoy it equally during their joint lives; if A dies first, then B and C are to enjoy it equally during their joint lives; but if B dies first, then A is to enjoy the whole during the joint lives of A, and C: To find the value of their respective shares.

SOLUTION.

The expectation of A on what he may happen to receive at the end of any one year may be considered in two parts, as depending on so many different events:

1^o A and B may be both alive, the probability of

which at the end of the first year is $\frac{a'b'}{ab}$, in which case he will receive $\frac{1}{2}$ the annuity: 2° A and C may be living and B dead, the probability of which at the end of the first year is $\frac{a'c'}{ac} \times (1 - \frac{b'}{b})$, in which case he will come in for the whole annuity. The sum of these two values, multiplied by $(1 + \rho)^{-1}$, will give $(1 + \rho)^{-1} (\frac{a'b'}{2ab} + \frac{a'c'}{ac} - \frac{a'b'c'}{abc})$ for the total value of the expectation of A on what he may happen to receive at the end of the first year.

By a similar process we may find the value of his expectation for the second, third, and every subsequent year to the utmost extent of human life: the sum of all which terms will be the total value of his interest in the annuity; and will be found equal to $\frac{1}{2}AB + AC - ABC$: whence the following rule.

§ 124. *Add half the value of an annuity on the joint lives AB, to the value of an annuity on the joint lives AC; from the sum, subtract the value of an annuity on the joint lives ABC: the remainder will be the value of the share of A.*

§ 125. The expectation of B on what he may happen to receive at the end of any one year may also be considered in two parts as depending on so many different events: 1° A and B may be both alive, the probability of which at the end of the first year is

$\frac{a'b}{ab}$, in which case he will receive $\frac{1}{2}$ the annuity: 2° B and C may be alive and A dead, the probability of which at the end of the first year is $\frac{b'c}{bc} \times (1 - \frac{a}{a'})$, in which case likewise he will receive $\frac{1}{2}$ the annuity. The sum of these two values therefore being multiplied by $(1+\rho)^{-1}$ will give $(1+\rho)^{-1} \times (\frac{a'b}{2ab} + \frac{b'c}{2bc} - \frac{a'b'c}{2a'bc})$ for the expectation of B on what he may happen to receive at the end of the first year.

By a similar process we may find the value of his expectation on what he may happen to receive at the end of the second, third, and every subsequent year, to the utmost extent of human life: the sum of all which terms will be the total value of his interest in the annuity; and will be found equal to $\frac{1}{2}(AB+BC-ABC)$: whence the following rule.

§ 126. *Subtract the value of an annuity on the three joint lives ABC, from the sum of the values of an annuity on each of the joint lives AB and BC: half the remainder will be the value of the share of B.**

§ 127. With respect to the share of C, it is solely equal to half the value of a reversionary annuity on the joint lives BC, after the life of A; which reversionary annuity is by the Scholium in page 70, ex-

* The value of B's interest in this annuity is the same as that in the preceding problem.

pressed by $\frac{1}{2}(BC - ABC)$: whence the following rule.

§ 128. *Subtract the value of an annuity on the joint lives ABC, from the value of an annuity on the joint lives BC: half the remainder will be the value of the share of C.*

§ 129. *Example.* Suppose the three lives to be aged 20, 30, and 40; interest 4 per cent, and the probabilities of living as at *Northampton*. The value of the share of each person will be as under: viz. that of

$$A = \frac{1}{2} \times 11.873 + 10.924 - 8.986 = 7.874.$$

$$B = \frac{1}{2}(11.873 + 10.490 - 8.986) = 6.689$$

$$C = \frac{1}{2}(10.490 - 8.986) \dots = .752$$

and the sum of all these three values will be the total value given for the purchase.

PROBLEM XVI.*

§ 130. D and his heirs, as soon as any *two* of three given lives A, B, C, become extinct, are to enter upon an annuity in order to enjoy the same during the life of the survivor: To find the value of his interest therein.

SOLUTION.

This annuity is evidently to continue during the longest of the three given lives; and the value of the

* Simpson's *Sup. Prob.* 20. Dodson, vol. iii. Ques. 79.

same without any restriction would be (by Prob. II) equal to $A+B+C-AB-AC-BC+ABC$: but since D (or his heirs) is not to receive any thing during the continuance of any two of these lives, the value of an annuity depending on this contingency must be subtracted from the preceding expression; and which value is (by Prob. III) denoted by $AB+AC+BC-2ABC$. Therefore $A+B+C-2AB-2AC-2BC+3ABC$ will be the value required: whence the following rule.

§ 131. *From the sum of the values of an annuity on each of the single lives, subtract twice the sum of the values of an annuity on each pair of joint lives; to the remainder add thrice the value of an annuity on the three joint lives: the sum will be the interest of D in this annuity, or the value of the reversion required.*

§ 132. *Example.* An estate is held on three lives whose ages are 20, 30, and 40, the income of which, as soon as any two of these lives become extinct, is to belong to D and his heirs during the continuance of the third life: what is the interest of D in this lease, reckoning the probabilities of living as at Northampton, and the rate of interest 4 per cent?

The sum of the values of an annuity on each of the single lives is $16.033 + 14.781 + 13.197 = 44.011$; the sum of the values of an annuity on each pair of joint lives is $11.873 + 10.924 + 10.490 = 33.287$;

and the value of an annuity on the three joint lives is 8.986. Therefore $44.011 - 2 \times 33.287 + 3 \times 8.986 = 4.395$ will be the value required.

PROBLEM XVII.*

§ 133. D and his heirs, as soon as any *one* of three given lives, A, B, C, becomes extinct, are to enter upon an annuity in order to enjoy the same during the continuance of the longest of the remaining lives: To find the value of his interest therein.

SOLUTION.

It is evident in this case also that the annuity is to continue during the longest of the three given lives; and such would be the value of it to D (or his heirs) were he to enter upon it immediately: but since he is not to receive any thing during the continuance of all the joint lives, the value of an annuity on those joint lives must be subtracted from the value of an annuity on the longest of the three lives, in order to obtain the required value. Wherefore, $A+B+C-AB-AC-BC+ABC-ABC=A+B+C-AB-AC-BC$ will be the value of his interest in the annuity: whence the following rule.

§ 134. *From the sum of the values of an annuity on each of the single lives, subtract the sum of the*

* Dodson, vol. iii, Ques. 78.

values of an annuity on each pair of joint lives: the remainder will be the interest of D in this annuity, or the value of the reversion required.

§ 135. *Example.* Suppose the lives to be aged 20, 30 and 40; the rate of interest 4 per cent, and the probabilities of living as at *Northampton*. In which case the value required would be $16.033 + 14.781 + 13.197 - (11.873 + 10.924 + 10.490) = 10.724$.

COROLLARY.

§ 136. Had two lives only, A and B, been concerned in this question, the value of the interest of D in the annuity would (by the same method of reasoning) become equal to $A + B - 2AB$: whence the following rule.

§ 137. *From the sum of the values of an annuity on each single life, subtract twice the value of an annuity on the two joint lives: the remainder will be the value required.*

For examples of the use and application of this corollary, see Question XXI in Chapter XII.

PROBLEM XVIII.*

§ 138. D and his heirs, as soon as any *one* of three given lives A, B, C, becomes extinct, are to

* Dodson, vol. iii. Ques. 80.

enter upon an annuity in order to enjoy the same during the joint continuance of the survivors: To find the value of his interest therein.

SOLUTION.

In this case, the annuity is to continue only as long as any two of the given lives are in being; and such would be the value of it to D, were he to enter on it immediately: but since he is to receive nothing during the joint continuance of all the lives, the value of an annuity on those joint lives must be subtracted from the value of an annuity to continue as long as any two of them are in being together. Therefore $AB + AC + BC - 2ABC - ABC = AB + AC + BC - 3ABC$, will be the value required: whence the following rule.

§ 139. *From the sum of the values of an annuity on each pair of joint lives, subtract thrice the value of an annuity on the three joint lives: the remainder will be the value required.*

§ 140. *Example.* Suppose the lives to be aged 20, 30 and 40; the rate of interest 4 per cent, and the probabilities of living as at *Northampton*. In which case the required value would be $11.873 + 10.924 + 10.490 - 3 \times 8.986 = 6.329$.

SCHOLIUM.

§ 141. The preceding problems contain some of the most general cases of survivorship, where not more than three lives are concerned; they are by no means offered as the whole which may arise, since the conditions of the problems are capable of infinite variety: but, by an attention to the method of deducing the values in the cases here given, it is presumed that no difficulty will be found in investigating any other case which may present itself.

§ 142. I would here observe, however, that the values, in all these problems, are deduced from the *whole continuance* of the lives on which the annuities depend: but if, from the conditions of the problem, the contingencies are *Deferred or Temporary*, then the annuities which result therefrom will be affected in a similar manner. In such cases, therefore, we must substitute the value of such deferred or temporary annuities, instead of annuities on the whole continuance of the lives, and proceed with such substituted values in the same manner as with the whole values of the annuities: agreeably to what I have already mentioned in Prob. II, cor. 2 and 4: and cor. 1 to the Scholium in page 70.

CHAPTER V.

*On Reversionary Annuities depending upon
a particular Order of Survivorship.*

§ 143. **A**MONG the cases of reversionary annuities, involving not more than three lives, and which are given in the Scholium at the end of the third Chapter,* there are two which oftentimes involve a contingency for which it is extremely difficult to find a correct algebraical solution fit for general use. These are the second and fourth cases there mentioned; and the contingency, alluded to, is that one of the two lives *in particular*, P or Q, shall die before or after the other.

As the investigations necessary to the solution of these cases will be of considerable importance in the further progress of this work, I shall devote a separate chapter to the examination of the subject. The solution of the cases themselves will lead to that of some other questions in Survivorships which, for want thereof, could not be inserted in the preceding chapter.

§ 144. Instead of the lives A, P, Q, as given in the scholium above mentioned, let the lives be A, B,

* See page 71.

C; of which, the lives B and C are supposed to be *in possession*, and the life A *in reversion*. The value of a reversionary annuity on the life A after the joint lives B C, is equal to $A - ABC$: but if A is not to enjoy the annuity unless B be the *first* that dies, of the two joint lives B C, it is evident that the value of the annuity will be considerably affected by this restriction.

In like manner, the value of a reversionary annuity on the life A after the longest of two lives B and C, is equal to $A - AB - AC + ABC$: but if A is not to enjoy the annuity unless B be the *last* that dies, of the two lives B C, it is evident the value of the annuity will in this case also be considerably affected by this restriction.

When the two lives in possession are of the same age, or nearly so, it will be an *equal* chance which of them dies before or after the other: but, as this is not always the case, it would be extremely desirable to obtain an expression, fit for general use, which should express this probability *for every year of human life*; because the expectation of receiving *each year's* rent will depend on such contingency. Unfortunately, however, this probability cannot be denoted by a *constant* quantity till after the extinction of the oldest of the two lives; because the chance of survivorship varies in every succeeding year till that period: and even then we can only approximate to the true value. When the difference between the ages of the lives in possession is not very great, the error will be but tri-

fling (as I have just observed) by supposing it to be an equal chance which of those lives dies before or after the other, during the probable time of their joint continuance; but when there is any considerable difference between the ages of the two lives, this cannot be an accurate method of proceeding. After that period, however, the contingency may in all cases be expressed with a tolerable degree of correctness. But, previous to the investigation of any of the problems connected with this subject, it will be necessary to premise the following lemmata.

LEMMA I.*

§ 145. To determine the probability that, out of two given lives A and B, one of them in particular, A, shall fail *before* the other.

SOLUTION.

It is evident that this circumstance may take place in any one year, 1° by A dying in that year and B surviving it; 2° by the extinction of both the lives in the year, A having died first. The probability of the first event happening in the first year is, by § 24 and 25, $\frac{(a-a')}{a} \times \frac{b'}{b}$; and the probability of the second event

* Phil. Trans. 1788, part II, page 332. Dodson, vol. iii. Ques. 21. Simpson, page 100. De Moivre, Prob. 18.

happening in the same period is $\frac{(a-a')}{a} \times \frac{(b-b')}{b} \times \frac{1}{2}$;
 the sum of these two, therefore, or $\frac{(a-a') \times (b+b')}{2ab}$, will
 denote the whole probability that A dies the first of
 the two lives in the *first* year.

In like manner, the probability of the first event
 happening in the second year is $\frac{(a-a'')b}{ab}$; and the pro-
 bability of the second event happening in the same
 period is $\frac{(a-a'') \times (b-b'')}{2ab}$; therefore the sum of these

* In all these annual contingencies I have supposed it to be an
equal chance that A dies before B, whatever may be the difference
 between the two ages: and though this is not strictly correct,
 except in those cases where the two lives are equal; yet as such
 chance is in the present investigation confined to *each particular*
year, and, when applied practically, is involved with other quan-
 tities which very much diminish the error that would otherwise
 arise from this hypothesis, it would be needless to render the so-
 lution more complicated and intricate by aiming at greater exact-
 ness. It will however appear from this remark that the result,
 here obtained, is only an *approximation* to the true value: which
 is more and more correct, the farther we continue the series.

When A is the youngest of the two lives, the fraction $\frac{1}{2}$ will
 exceed the chance that A dies before B; and consequently the
 value of ψ , as found by the lemma, will *exceed* the true value of
 the probability that A dies before B in any part of their co-
 existence. On the other hand, when A is the oldest of the two
 lives, the chance that A dies before B will be more than $\frac{1}{2}$, and
 consequently ψ will in this case *fall short* of the true value of the
 probability that A dies before B in such period. Proper al-
 lowances may be made in such cases, when there is any material
 difference in the ages of the two lives.

two, or $\frac{(a'-a'') \times (b'+b'')}{2ab}$, will denote the whole probability that A dies the first of the two lives in the *second* year. And, if this value be added to the one just found for the first year, it will give $\frac{(a-a') \times (b+b')}{2ab} + \frac{(a'-a'') \times (b'+b'')}{2ab}$ for the probability that A dies the first of the two lives in *two* years.

By a similar method of proceeding it will be found that $\frac{(a''-a''') \times (b''+b''')}{2ab}$ will denote the whole probability that A dies the first of the two lives in the *third* year: which, being added to the value just found for two years, will give $\frac{(a-a') \times (b+b')}{2ab} + \frac{(a'-a'') \times (b'+b'')}{2ab} + \frac{(a''-a''') \times (b''+b''')}{2ab}$ for the probability that this event will happen in *three* years.

And so on with respect to the values for all the subsequent years to the utmost extent of human life: and, if we make n equal to the difference between the age of the oldest of the given lives and that age in the table of observations at which human life becomes

extinct, we shall find that the series $\frac{(a-a') \times (b+b')}{2ab} + \frac{(a'-a'') \times (b'+b'')}{2ab} + \frac{(a''-a''') \times (b''+b''')}{2ab} + \dots + \frac{(a^{(n)}-a^{(n+1)}) \times (b^{(n)}+b^{(n+1)})}{2ab}$

will express the whole chance of A dying before B, during that period. But this series, reduced to more

simple terms, is equal to $\frac{1}{2ab} \times [(a - a') \times (b + b') + (a' - a'') \times (b' + b'') + (a'' - a''') \times (b'' + b''') \dots (\alpha - \alpha) \times (\beta + \beta)]$, the sum of which I shall denote by ψ ; therefore ψ will express the probability that A will die before B during the probable term of their joint continuance: and its application will be shown in the solution of the subsequent problems.

§ 146. In finding the numerical value of this series for lives of different ages, but whose common difference of age is the same, it will be most convenient to begin with the last term of it, or $(\alpha - \alpha) \times (\beta + \beta)$; and to proceed upwards till we come to the ages of the given lives, as in finding the numerical value of Annuities: see Prob. I, cor. 2. By this method, the process for finding the probability of survivorship between two other lives, whose ages have the same common difference as the given lives, will be considerably abridged; since the latter terms of the series will be common to both cases.

§ 147. *Example.* Let it be required to find the probability that A the youngest of two lives, whose difference of age is 10 years, shall die before B the eldest; according to the probabilities of living as observed in Sweden, amongst males and females collectively. The process for finding such values, for every year of human life, will best appear from the following specimen.

Age of A	Age of B	
86	96	$\frac{1}{2 \times 144} \times 35 \dots \dots \dots = 1215$
85	95	$\frac{1}{2 \times 2 \times 189} \times [45 \times (2+1) + 35] \dots \dots \dots = 2248$
84	94	$\frac{1}{2 \times 5 \times 244} \times [55 \times (5+2) + 45 \times (2+1) + 35] \dots \dots \dots = 2275$
83	93	$\frac{1}{2 \times 11 \times 309} \times [65 \times (11+5) + 55 \times (5+2) + 45 \times (2+1) + 35] \dots \dots \dots = 2346$
82	92	$\frac{1}{2 \times 21 \times 384} \times [75 \times (21+11) + 65 \times (11+5) + 55 \times (5+2) + 45] \dots \dots \dots = 2477$
81	91	$\frac{1}{2 \times 33 \times 468} \times [84 \times (33+21) + 75 \times (21+11) + 65 \times (11+5) + 45] \dots \dots \dots = 2762$
80	90	$\frac{1}{2 \times 47 \times 558} \times [90 \times (47+33) + 84 \times (33+21) + 75 \times (21+11) + 45] \dots \dots \dots = 2999$
&c	&c	&c &c &c

By these examples it will be seen that the probability, that a life aged 80 will die before another life aged 90, is expressed by the fraction $\cdot 2999$; where *certainty* is denoted by unity: and that, if the two lives were 82 and 92, the probability would be expressed by $\cdot 2477$. It will be seen also, on inspection, that each year's series is for the most part composed of the terms which form the preceding series: and consequently that the finding the probability of survivorship between lives of *several* ages, whose common difference is the same, is not much more laborious than finding it for the youngest of those lives.

§ 148. The above example relates to the probabilities of survivorship between the two lives as observed in Sweden amongst mankind *at large*: but if one of those lives be a *male* and the other a *female* the results will be materially different, as may be seen by the two following tables, where the answer will vary according as the male or the female is the oldest of the two lives.

Age of A Male.	Age of B Female.	Probability that A dies first.
87	97	$\cdot 1209$
86	96	$\cdot 1956$
85	95	$\cdot 2405$
84	94	$\cdot 2811$
83	93	$\cdot 3147$
82	92	$\cdot 3342$
81	91	$\cdot 3478$
80	90	$\cdot 3534$

Age of A Female.	Age of B Male.	Probability that A dies first.
87	97	$\cdot 0000$
86	96	$\cdot 0000$
85	95	$\cdot 1228$
84	94	$\cdot 1793$
83	93	$\cdot 2043$
82	92	$\cdot 2437$
81	91	$\cdot 2678$
80	90	$\cdot 2797$

Two other tables likewise might be formed for the same ages; viz. for those cases where *both* the lives are males, and for those cases where *both* the lives are females: and the results would in these cases also be different from those above adduced. But enough has been said to enable the reader to calculate the true probability of survivorship according to any case that may come before him.*

* Mr. Morgan has given a "Table shewing the probabilities of survivorship between two persons of all ages, whose common difference of age is not less than 10 years, computed from the *Northampton* table of observations." See *Phil. Trans.* vol. lxxviii. p. 337: or Dr. Price's *Obs. on Rev. Pay.* vol. i. p. 406. He says that the probabilities are very nearly the same from whatever table of observations they are computed: the *accuracy* of which remark may be best seen by a comparison with the examples above given.

The table, here alluded to, is the only one hitherto calculated for showing the probabilities of survivorship between any two lives. As that table however is applicable only to the valuation of annuities as deduced from the *Northampton* observations, and is moreover adapted to every decade only of human life, I shall point out an easy method whereby we may obtain a *near* value of such probability; which, though not strictly true, will be sufficiently correct if the two lives are between 10 and 70 years of age: at least, till we are possessed of more enlarged tables on this subject. Let the expectation of A's life, as found by Prob. I, cor. 7, be denoted by e ; and let the expectation of B's life as found in the same manner be denoted by ε : then will $\psi = \frac{e}{2e}$.

This formula is deduced from the assumption that the decrements of life are equal and uniform at every age, agreeably to the principles of M. De Moivre which will be more fully explained in Chapter IX: and is one of the many instances of the utility and convenience of that celebrated hypothesis. The following ex-

COROLLARY I.

§ 149. If the sum of any number of terms of that series be subtracted from the probability of the failing of the joint lives in that term, the difference will denote the probability that B will die before A in that time. Consequently (since it is certain that one or other of them will die before the end of n years) if the sum of the whole series be subtracted from unity, the difference, or $1 - \psi$, will express the whole chance of B dying before A during the probable term of their joint continuance.

COROLLARY II.

§ 150. When the two lives are equal, the sum of the general series given in the problem is equal to $\frac{1}{2}$: that is, $\psi = \frac{1}{2}$.

amples, deduced from the *Northampton* observations and compared with the values found by the Lemma, will show how far it may be relied on.

Age of A.	Age of B.	Value by Hypothesis.	Value by Lemma.
20	30	·4228	·4231
20	40	·3452	·3429
20	50	·2692	·2650
20	60	·1976	·1931
30	40	·4082	·4094
30	50	·3182	·3170
30	60	·2336	·2302
40	50	·3897	·3938
40	60	·2862	·2882
50	60	·3671	·3756

SCHOLIUM.

§ 151. Since A and B may denote lives of any ages, and therefore the series given in the problem extends to all cases; whereby ψ becomes a general expression for the probability of survivorship between any two lives: yet, as different probabilities may hereafter arise in the same problem, and as much confusion may be created by the use of the same character to express different quantities, I here take the opportunity of observing that I shall denote the several probabilities of survivorship that may take place between any two of the three lives A, B, C, by the following symbols: viz.

$$A \text{ dying before } B = \psi$$

$$A \text{ } C = \varpi$$

$$B \text{ } C = \phi$$

whence it follows, by Cor. 1, that the probability of

$$B \text{ dying before } A = 1 - \psi$$

$$C \text{ } A = 1 - \varpi$$

$$C \text{ } B = 1 - \phi$$

LEMMA II.*

§ 152. To determine the probability that, out of two given lives A and B, one of them in particular, A, shall die *after* the other.

* Phil. Trans. 1794, part II, page 224. Dodson, vol. iii. Ques. 19.

SOLUTION.

It is manifest that this circumstance can take place in the first year only by the extinction of both the lives, A having died last; the probability of which is $\frac{(a-a')}{a} \times \frac{(b-b')}{b} \times \frac{1}{2}$,* an expression which, for reasons that will hereafter appear, I shall make equal to $\frac{a-a'}{a} - \frac{(a-a') \times (b+b')}{2ab}$.

But in the second and following years, the event may take place, 1° by the extinction of both the lives in the year, A having died last; 2° by the decease of A in the year, B having died in either of the preceding years. The probability of the first of these events happening in the second year is $\frac{(a-a'') \times (b-b')}{2ab}$, and the probability of the second event happening in the same period is $\frac{(a-a'')}{a} \times (1 - \frac{b}{b})$: the sum of these two, therefore, or $\frac{a-a''}{a} - \frac{(a-a'') \times (b+b')}{2ab}$, will denote the whole probability that A dies after B in the second year; and which expression, being added to the value just found for the first year, will give $\frac{a-a'}{a} - \frac{(a-a') \times (b+b')}{2ab} + \frac{a-a''}{a} - \frac{(a-a'') \times (b+b')}{2ab}$, for the probability that A dies after B in two years.

* See the note in page 109.

In like manner, the probability of the first event happening in the third year is $\frac{(a''-a''') \times (b''-b''')}{2ab}$, and the probability of the second event happening in the same period is $\frac{(a''-a''')}{a} \times (1 - \frac{b''}{b})$; the sum of which, or $\frac{a''-a'''}{a} - \frac{(a''-a''') \times (b''+b''')}{2ab}$, will denote the whole probability of A dying after B in the *third* year: and which expression, being added to the value just found for two years, will give $\frac{a-a'}{a} - \frac{(a-a') \times (b+b')}{2ab} + \frac{a'-a''}{a} - \frac{(a'-a'') \times (b'+b'')}{2ab} + \frac{a''-a'''}{a} - \frac{(a''-a''') \times (b''+b''')}{2ab}$, for the probability that A dies after B in *three* years.

And so on with respect to the values for every subsequent year: and if we make n equal to the difference between the age of the oldest of the given lives and that age in the table of observations *at which human life becomes extinct**, we shall find that the

* Mr. Morgan has taken this term equal only to the difference between the age of the oldest of the given lives and the age of *the oldest life* in the table of observations: whereby he has omitted the chance of one life dying after the other in the *last* year of their joint existence. The reader must bear this in mind when comparing the values, deduced by this rule, with the values deduced from the rule given by Mr. Morgan. I would here observe, that the formula above given is not only the most correct of the two, but also renders the several expressions, in which it is afterwards used, much more simple than those which are deduced according to Mr. Morgan's assumption. And, agreeably to these principles,

series $\frac{a-a'}{a} + \frac{a'-a''}{a} + \frac{a''-a'''}{a} + \dots \dots \dots \frac{a-a}{a} -$
 $\left[\frac{(a-a') \times (b+b')}{2ab} + \frac{(a'-a'') \times (b'+b'')}{2ab} + \frac{(a''-a''') \times (b''+b''')}{2ab} + \dots \right.$
 $\left. \dots \dots \frac{(a-a) \times (b+b)}{2ab} \right]$ will express the whole chance

of A dying after B *during that period*; that is, during the probable term of their joint continuance.

But the first part of this series, or $\frac{a-a'}{a} + \frac{a'-a''}{a} + \frac{a''-a'''}{a} + \dots \dots \dots \frac{a-a}{a}$, is evidently equal to $\frac{a-a}{a} = (1 - \frac{a}{a})$; that is, equal to the probability that the life A shall fail in that period: and the latter part of the series, which is to be subtracted, is (by the preceding Lemma) equal to ψ : consequently the total required value for n years will be denoted by $(1 - \frac{a}{a}) - \psi$.

it will be found that the table which Mr. Morgan has given, of the probabilities of one life *dying after* the other, is totally useless: since its application may always be avoided in practice; as will be evident from the method pursued in the investigation of the problems in this, and in the eighth Chapter of the present work.

Should the reader be desirous of knowing the probability that A will die after B during the period of their joint continuance, according to Mr. Morgan's assumption, he will find that it is denoted by $(1 - \frac{a}{a}) - \psi$: that is, by substituting a for a in the formula which I have given in the text. In like manner the probability of B dying after A in that period will, upon the same assumption, be denoted by $\psi - \frac{\beta}{b}$: that is, by substituting β for β in the formula given in the scholium. And so of the other quantities there adduced.

COROLLARY I.

§ 153. If the sum of any number of terms in the above series be subtracted from the probability of both the lives failing in that time, the difference will denote the probability that B will die after A during that period. Consequently if the sum of the whole series above given be subtracted from $(1 - \frac{a}{a}) \times (1 - \frac{\beta}{b}) = 1 - \frac{a}{a} - \frac{\beta}{b}$,* (or the probability that both the lives fail in that period) the difference, or $(1 - \frac{a}{a} - \frac{\beta}{b}) - (1 - \frac{a}{a} - \psi) = \psi - \frac{\beta}{b}$, will express the chance of B dying after A in that period.

COROLLARY II.

§ 154. When the two lives are both of the same age, ψ becomes equal to $\frac{1}{2}$; as already mentioned in the second corollary to the preceding Lemma: consequently the sum of the series in the present Lemma becomes also equal to $\frac{1}{2}$.

SCHOLIUM.

§ 155. In order to prevent the confusion of quantities alluded to in the Scholium given in page 116, I shall here denote the several probabilities of survivor-

* Since it is *certain* that one or other of the lives will be extinct at the end of n years, it follows that the quantity $\frac{a\beta}{ab}$, which arises from this product, will vanish altogether.

ship, which have been the subject of this Lemma, and which may arise between any two of three lives A, B, C, by the following symbols: viz.

$$A \text{ dying after } B = 1 - \psi - \frac{\alpha}{a}$$

$$A C = 1 - \varpi - \frac{\alpha}{a}$$

$$B C = 1 - \phi - \frac{\beta}{b}$$

whence it follows by Cor. 1, that the probability of

$$B \text{ dying after } A = \psi - \frac{\beta}{b}$$

$$C A = \varpi - \frac{\gamma}{c}$$

$$C B = \phi - \frac{\gamma}{c}$$

§ 156. It should here be remarked that the above expressions denote the respective probabilities of survivorship for n years only, or during the probable time of the joint continuance of the two lives; and that the values are deduced without any regard to seniority. Therefore, when A is the oldest of the two lives the general expression in the Lemma will become $1 - \psi$; because α becomes equal to 0, and consequently the fraction $\frac{\alpha}{a}$ vanishes altogether. But, when A is the youngest of the two lives, that expression will not denote the *whole* probability of A dying after B, since there is a further chance of A dying, after having survived B. In order to determine this probability for the subsequent years, the series in the Lemma must be continued till the extinction of A's life: whence it

will be found that the probability of A dying after B in $(n+1)$, $(n+2)$, $(n+3)$, &c, years will be respectively denoted by $1 - \psi - \frac{a}{a}$, $1 - \psi - \frac{a}{a}$, $1 - \psi - \frac{a}{a}$, &c; to the utmost extent of A's life, at which period the expression becomes $1 - \psi$.

In like manner, when B is the oldest of the two lives, the probability that B will die after A becomes equal to ψ : but, if B be younger than A, the general expression above given, $\psi - \frac{\beta}{b}$, denotes the probability of that event taking place during n years only, or during the probable time of their joint continuance. And the probability of the same event taking place in $(n+1)$, $(n+2)$, $(n+3)$, &c, years will be respectively denoted by $\psi - \frac{\beta}{b}$, $\psi - \frac{\beta}{b}$, $\psi - \frac{\beta}{b}$, &c; to the utmost extent of B's life, when the expression at length becomes equal to ψ . The same observations will apply to the other quantities above given.

PROBLEM XIX.*

§ 157. To find the value of a reversionary annuity on the life A, after the longest of two lives B and C, on condition that B dies *after* C.

* Simpson's *Sup. Prob.* 34. Dodson, vol. iii. Ques. 30. Morgan, *Prob.* 27, cor. and in *Phil. Trans.* for 1794, page 240.

SOLUTION.

The chance which A has of receiving this annuity at the end of any one year will depend on the continuance of his life to the end of such term, and on the extinction of both the lives B and C previous thereto; restrained however to the contingency that B dies last. It is this contingency which it is so difficult to represent in such a manner as to be generally useful. In the short space of one year, as I have before observed (page 109), the error is not material by taking one *half* the product of the probabilities that the two lives shall fail in that period. “But, when
 “the number of years and the difference between the
 “ages of the two lives are considerable, those chances
 “must vary in proportion; and therefore unless the
 “contingency is blended with another which shall
 “very much diminish the probability of the event,
 “the solution, by thus indiscriminately supposing the
 “chances to be equal, must be rendered *extremely*
 “inaccurate.”*

§ 158. If the probability of B dying after C in *one, two, three, &c.* years (as found by the second Lemma) be severally denoted by $y', y'', y''', \&c.$ the ex-

* These are Mr. Morgan's own words when speaking of Mr. Simpson's method of solution: see *Phil. Trans.* 1791, page 276. We shall find, in the sequel, that he has fallen into the same error himself.

pectation which A has of enjoying the annuity at the end of those years will be represented, with a sufficient degree of accuracy,* by $\frac{a.y'}{a(1+\rho)}$, $\frac{a.y''}{a(1+\rho)^2}$, $\frac{a.y'''}{a(1+\rho)^3}$, &c, respectively. But, in this case the true value of the reversionary annuity could not be expressed by less than n different series; and therefore would be wholly unfit for general use.

§ 159. It appears, from what has been here said, that the chance of one life dying before or after another differs in *every* year of their joint existence; and that it is not capable of being represented by a *constant* quantity till the extinction of the oldest of such lives. *After* that period, however, the expectation which A has of enjoying the annuity at the end of any subsequent year may be determined sufficiently near for any useful purpose by the help of the preceding lemmata. Consequently, all that appears further ne-

* Mr. Morgan says (*Phil. Trans.* for 1794, p. 238) that these quantities would give the *exact* value of the reversionary annuity: but he asserts this on the presumption that the values deduced from the Lemma give the *true* probabilities of survivorship for *every* year of human life. Whereas those values approximate only in proportion to the length of the series; and are incorrect in the first terms of such series, when there is any considerable inequality between the ages of the two lives. Nevertheless, if they could be at all rendered fit for practical purposes, they would enable us to approximate more nearly to the true value of the reversionary annuity than the inaccurate method hitherto adopted.

cessary for the proper solution of the problem is such an expression as will approximate to the value of the chance that B will die after C during the several years of their *joint* continuance. Let such expression be denoted by χ ; (the value of which will be the subject of a future inquiry, see § 173): then will the value of the reversionary annuity, depending on the contingency mentioned in the problem, be determined in the following manner.

§ 160. It is manifest that the payment of the annuity in any one year depends on the continuance of the life A to the end of that year, and on the extinction of both the lives B and C previous thereto, B having died last: the probabilities of which events for the first, second, third, &c. years are respectively denoted by $\frac{a'}{a}(1-\frac{b'}{b}) \times (1-\frac{c'}{c})\chi$, $\frac{a''}{a}(1-\frac{b''}{b}) \times (1-\frac{c''}{c})\chi$, $\frac{a'''}{a}(1-\frac{b'''}{b}) \times (1-\frac{c'''}{c})\chi$, &c. Consequently, the sum of the expectations of receiving the annuity at the end of those periods respectively will, for the first n years,* be denoted by the following series,

* I would here observe, once for all, that in this and the two subsequent problems I take n to denote the number of years between the age of the oldest life involved in the question, and that age in the table of observations when human life becomes extinct: consequently the value of n will vary according to the three cases given in these problems.

$$\begin{aligned}
 & (1+p)^{-1} \times \chi \left[\frac{a'}{a} - \frac{a'b'}{ab} - \frac{a'c'}{ac} + \frac{a'b'c'}{abc} \right] + \\
 & (1+p)^{-2} \times \chi \left[\frac{a''}{a} - \frac{a''b''}{ab} - \frac{a''c''}{ac} + \frac{a''b''c''}{abc} \right] + \\
 & (1+p)^{-3} \times \chi \left[\frac{a'''}{a} - \frac{a'''b'''}{ab} - \frac{a'''c'''}{ac} + \frac{a'''b'''c'''}{abc} \right] + \\
 & \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
 & (1+p)^{-n} \times \chi \left[\frac{a}{a} - \frac{a\beta}{ab} - \frac{a\gamma}{ac} + \frac{a\beta\gamma}{abc} \right].
 \end{aligned}$$

§ 161. *Case 1.* Let A be the oldest of the three lives. It is evident that in this case the series above given will denote the whole value of the reversionary annuity required: because in the n^{th} year the life A becomes extinct, and all the subsequent terms of the series vanish. But the above series is equal to $\chi[A - AB - AC + ABC]$: that is, equal to the value of a reversionary annuity on the life A after the longest of two lives B and C, multiplied by the chance that B dies after C.*

* According to the method of solution adopted by Mr. Simpson and Mr. Morgan, the values (in all those cases where A is the oldest life) will be precisely the same, whatever be the difference of age between B and C, whether the contingency depends on B dying after C, or C dying after B. Thus, the value of an annuity on a life aged 78 to be entered upon at the extinction of two lives aged 15 and 75 (which is one of the cases given by Mr. Morgan) is the same whether the contingency depends on the younger life dying after the elder, or on the elder dying after the younger. But it must be evident that an annuity depending on the former contingency is worth more than a similar annuity depending on the latter contingency. For, the time of A's coming into possession is the

§ 162. *Case 2.* Let B be the oldest of the three lives. In this case, the above series will denote the

same in both cases; viz. on the extinction of *both* the lives: therefore the value of the annuity will be affected only by the contingency of one life dying after the other.

Mr. Morgan, in attempting to give a more correct solution for the value of such annuities *after* the extinction of the oldest life, has overlooked the most material part of the process, which is to obtain a more accurate expression for the value of such annuities *during the joint continuance* of all the lives. I am aware that Mr. Morgan asserts that, by taking it as an *equal* chance whether B dies before or after C in any given period, the value of the annuity which results from this assumption will be sufficiently near the *true* value for any useful purpose: and he has given some examples with a view to prove the accuracy of his remark. I shall here, however, take the opportunity of observing that what he calls the *true* value is only an *approximation*, which differs most from the true value in those very cases where it is most wanted as a test. He has deduced certain values from a false hypothesis; and afterwards, assuming these values as if *correct*, endeavours to prove that another method of approximation used by him is accurate because it agrees with these assumed values. Now, from what has been said in the preceding pages, I think it must be evident that, when there is any considerable difference between the ages of the two lives, the value of the probability deduced by the Lemmata will not be the correct value for *every year* of human life: neither will the method of proceeding, alluded to in § 158, enable us in such cases to obtain the true value of the reversionary annuity.

These observations apply to the table inserted by Mr. Morgan in the *Phil. Trans.* for 1794; page 234, and to the examples given by him in page 239 of the same volume. Mr. Morgan says "that the approximations and exact values do not differ much from each other till the last years of the oldest life:" but the fact is that they differ nearly in the same ratio throughout the whole time of the joint continuance of the two lives. His own remarks prove the inaccuracy of his method of reasoning.

value of the reversionary annuity for the first n years (n now denoting the number of years between the age of B and that age in the table of observations when human life becomes extinct): and the sum of it will be found equal to $\propto [(A)^t - AB - (AC)^t + ABC]$. For, the first and third of these perpendicular series being continued to n terms only, the sum of those terms will, by Prob. I, cor. 4, be accurately represented by the characters here given: and the second and fourth perpendicular series evidently denote the *whole* value of the annuities on those lives respectively; since in the n^{th} year the life B becomes extinct and all the subsequent terms of those series vanish.

Now, in order to determine the expectation of receiving the several rents in the remaining years of A's life, it should be observed that the payment of the annuity in any one year depends on the continuance of the life A to the end of that year, and on the probability that B dies after C previous thereto: but the probability, that B will die after C in the $(n+1)^{\text{st}}$ and all the subsequent years, is (by the scholium to the second Lemma) denoted by the constant quantity $(1-\phi)^*$. Consequently the sum of the expectations of receiving the $(n+1)^{\text{st}}$, $(n+2)^{\text{nd}}$, $(n+3)^{\text{rd}}$, &c, year's rents will be expressed by the series $\frac{(1-\phi)a}{a(1+\phi)^{n+1}} +$

* Because, in the general expression there given, β becomes equal to 0 when B is the oldest life; and the fraction $\frac{\beta}{b}$ consequently vanishes.

$\frac{(1-\phi)\alpha''}{a(1+\phi)^{n+2}} + \frac{(1-\phi)\alpha'''}{a(1+\phi)^{n+3}} + \&c$: which, being continued for all the subsequent years of A's life, will show the true value of all the rents to be received after n years. But the sum of this series is, by Prob. I, cor. 3, equal to $(1-\phi) \times (A)^d$; and which, being added to the first n terms of the several collateral series above found, will make the total value of the annuity in this case equal to $\chi[(A)^t - AB - (AC)^t + ABC] + (1-\phi) \times (A)^d$. But, since $(A)^t = A - (A)^d$, and $(AC)^t = AC - (AC)^d$, as appears from Prob. I, cor. 4, it follows that this value may be more conveniently expressed by $\chi[A - AB - AC + ABC] + (1-\phi - \chi) \times (A)^d + \chi(AC)^d$.*

§ 163. *Case 3.* Let C be the oldest of the three lives. The value of all the rents for the first n years (n now denoting the number of years between the age of C and that age in the table of observations when human life becomes extinct) may in this case,

* I would here observe that the rule given by Mr. Simpson for the solution of this case is expressed by $(A - 2AB + ABB) \frac{s}{\eta^2}$: where s denotes the expectation of the life B as deduced from the rule in page 49, and where η denotes the expectation of the life C as deduced from the same rule.

When C is the oldest life, this value must be deducted from the whole value of an annuity on the life A after the longest of the two lives B and C; agreeably to the method which will be explained in the note in page 140.

as in the last, be denoted by the first n terms of the series given in the Problem; and the sum of which will now be equal to $x[(A)^t - (AB)^t - AC + ABC]$. For, the first and second of those perpendicular series being continued to n terms only, the sum of those terms will be truly denoted by the characters here given: but the last two perpendicular series evidently denote the value of annuities on the *whole continuance* of those lives respectively; because in the n^{th} year the life C is extinct.

Now in order to determine the expectation of receiving the several rents in the remaining years of A's life, it should be observed that the probability of B dying after C in $(n+1)$, $(n+2)$, $(n+3)$, &c, years is, by the scholium to the second Lemma, respectively denoted by $(1 - \phi - \frac{\beta}{b})$, $(1 - \phi - \frac{\beta}{b})$, $(1 - \phi - \frac{\beta}{b})$, &c: consequently, since the payment of the annuity in any one year depends on the continuance of A's life to the end of that year and on B having died after C previous thereto, it follows that if these values be severally multiplied by $\frac{\alpha}{a}$, $\frac{\alpha}{a}$, $\frac{\alpha}{a}$, &c, (or the probability that A will live $(n+1)$, $(n+2)$, $(n+3)$, &c, years respectively) the products thence arising will express the whole chance of receiving each year's rent respectively. And these products, being again multiplied by the present value of £1 due at the end of those respective periods, will give the expectations of receiving such sum at the end of those years: and which will be expressed by the following series,

$$\begin{aligned}
 & (1+\rho)^{-(n+1)} \times \left[(1-\phi) \frac{a'}{a} - \frac{a'\beta'}{ab} \right] + \\
 & (1+\rho)^{-(n+2)} \times \left[(1-\phi) \frac{a''}{a} - \frac{a''\beta''}{ab} \right] + \\
 & (1+\rho)^{-(n+3)} \times \left[(1-\phi) \frac{a'''}{a} - \frac{a'''\beta'''}{ab} \right] + \\
 & \quad \&c \qquad \qquad \&c \qquad \qquad \&c.
 \end{aligned}$$

But the sum of these two perpendicular series, continued to the utmost extent of A's life, is by Prob. I, cor. 3, equal to $(1-\phi) \times (A)^d - (AB)^d$; and which, being added to the first n terms of the several collateral series above found, will make the total value of the annuity in this case equal to $\chi \left[(A)^t - (AB)^t - AC + ABC \right] + (1-\phi-\chi) \times (A)^d - (1-\chi) \times (AB)^d$: an expression which, from what has been said in the last case, may be more conveniently denoted by $\chi \left[A - AB - AC + ABC \right] + (1-\phi-\chi) \times (A)^d - (1-\chi) \times (AB)^d$.

COROLLARY.

§ 164. If the two lives in possession are both of the same age, B, then ϕ and χ each become equal to $\frac{1}{2}$, and the quantities $(AB)^d$ and $(AC)^d$ vanish altogether: consequently the expression in each case becomes $\frac{1}{2} [A - 2AB + ABB]$; that is, equal to half the value of a reversionary annuity on the life A after the longest of two equal lives B and B.

But, if the two lives in possession are of the same age with the life *A* in reversion, this expression will become $\frac{1}{2}[A - 2AA + AAA]$.

PROBLEM XX.*

§ 165. To find the value of a reversionary annuity on the life *A*, after two joint lives *BC*, on condition that *B* dies *before C*.

SOLUTION.

The payment of this annuity at the end of any one year depends on the continuance of the life *A* to the end of that year, and on the extinction of the life *B* before *C* previous thereto. As this latter contingency cannot be conveniently represented with accuracy for every year of human life, it becomes necessary, as in the last problem, to have recourse to an approximation during the number of years that the lives have a chance of continuing together. Therefore let the same symbols be used as in that problem: and let us consider the payment of the annuity, during the first *n* years, as depending in each year on one or other of two events: 1° that *A* lives to the end of the year, and that *B* and *C* both fail previous thereto *B* having died first: 2° that only *B* dies previous thereto,

* Simpson's *Sup. Prob.* 35. Dodson, vol. iii. Ques. 49. Morgan, *Prob.* 27: and in *Phil. Trans.* for 1794, page 235.

and that A and C both live. The probability of the happening of the first event in the first year is $\frac{a}{a} (1 - \frac{b}{b}) \times (1 - \frac{c}{c}) \times (1 - \chi)$;* and the probability of the happening of the second event in the same period is $(1 - \frac{b}{b}) \times \frac{a}{a} \frac{c}{c}$: therefore the sum of these two, reduced to simple terms and multiplied by $(1 + \rho)^{-1}$, will give $(1 + \rho)^{-1} \times \left[\frac{a}{a} - \frac{a}{a} \frac{b}{b} - \left(\frac{a}{a} - \frac{a}{a} \frac{b}{b} - \frac{a}{a} \frac{c}{c} + \frac{a}{a} \frac{b}{b} \frac{c}{c} \right) \chi \right]$ for the total value of the expectation of receiving the first year's rent. By a similar method of reasoning we may find the expectation of receiving the second, third, &c, year's rents: and the sum of all these expectations for the first n years, will be denoted by the following series,

[illegible]

§ 166. *Casè. 1.* Let A be the oldest of the three lives. It is evident that in this case the series above

* Since χ denotes the chance that B will die *after* C in every year of their joint existence, it follows that $1 - \chi$ will denote the chance that B will die *before* C in the same periods.

given will express the whole value of the reversionary annuity required: because in the n^{th} year the life A becomes extinct, and all the subsequent terms of the series vanish. But the above series is in such case evidently equal to $A - AB - \chi[A - AB - AC + ABC]$.

§ 167. *Case 2.* Let B be the oldest of the three lives. In this case, the above series will denote the value of the reversionary annuity for the first n years, and the sum of it will be found to be $(A)^t - AB - \chi[(A)^t - AB - (AC)^t + ABC]$. In the subsequent years, the chance of receiving the annuity will depend on the continuance of the life A , and on the probability of B dying before C : therefore the value of the annuity for the remaining years will be denoted by the sum of the series $\frac{\phi \cdot \alpha^I}{a(1+\rho)^{n+1}} + \frac{\phi \cdot \alpha^{II}}{a(1+\rho)^{n+2}} + \frac{\phi \cdot \alpha^{III}}{a(1+\rho)^{n+3}} + \&c = \phi(A)^d$. Consequently the total value of the required annuity will be $A - AB - \chi[A - AB - AC + ABC] - (1 - \phi - \chi) \times (A)^d - \chi(AC)^d$.

§ 168. *Case 3.* Let C be the oldest of the three lives. In this case, the above series will also denote the value of the reversionary annuity for the first n years; the sum of which will now become equal to $(A)^t - (AB)^t - \chi[(A)^t - (AB)^t - AC + ABC]$:

and the value of the annuity for the remaining years will be truly expressed, as in the last case, by $\phi (A)^d$. Consequently the total value of the annuity required will be $A - AB - \chi [A - AB - AC + ABC] \rightarrow (1 - \phi - \chi) \times (A)^d + (1 - \chi) \times (AB)^d$.

COROLLARY.

§ 169. If the two lives in possession are both of the same age B, then ϕ and χ will each (as in the corollary to the last problem) become equal to $\frac{1}{2}$, and the quantities $(AB)^d$ and $(AC)^d$ vanish altogether: consequently the value of the reversionary annuity in each case is expressed by $\frac{1}{2}[A - ABB]$.

But when the two lives in possession are of the same age as the life A in reversion, this expression will become equal to $\frac{1}{2}[A - AAA]$.

SCHOLIUM.

§ 170. If the value found according to either of the three cases in this problem be added to the value found according to the corresponding case in the preceding problem, their sum will be equal to $A - AB$. Consequently, if the value of the annuity in either case of the preceding problem be once determined, the value of the same annuity according to the corresponding case in the present problem may be easily found (provided the ages of the three lives A, B, C,

are the same in both instances) by subtracting the former value from $A - AB$: and *vice versa*.

This is almost self-evident: for, taking the contingencies of both problems, it is manifest that the annuity is certain to be enjoyed by A after the decease of B, provided he lives so long. Consequently the sum of the values found by both problems must in each case be equal to the value of a reversionary annuity on the life A after the decease of B. The truth of this scholium is indeed evident on inspection,

PROBLEM XXI.*

§ 171. B and C possess an annuity between them; which, if B survives C, is afterwards to be equally divided between A and B during their joint existence, and then is to go entirely to the survivor for his life; To find the value of the share of A in this annuity.

SOLUTION.

Since A is not to receive any thing if C survives B, it is evident that he is entitled to half the value of a reversionary annuity on the joint lives AB after the extinction of the life C; and also to the whole value of a reversionary annuity on his own life after the extinction of the life B, provided B dies after C. The

* Simpson, Prob. 29, and *Sup.* Prob. 36.

former value is, by the scholium in page 70, equal to $\frac{1}{2}(AB - ABC)$; and the latter value may be found by Problem XIX, according as A, B, or C, is the oldest of the three lives: whence, the following values may be readily deduced, for the three cases there mentioned.

That is, when A is the oldest of the three lives, the required value will be equal to $\chi(A - AC) + (\frac{1}{2} - \chi) \times (AB - ABC)$.* When B is the oldest of the

* If we take χ to denote $\frac{1}{2}$ (that is, if we suppose with Mr. Morgan indiscriminately that it is an even chance whether B dies before or after C in any period of their joint existence, whatever be the difference of age between the two lives) then will this formula become equal to $\frac{1}{2}(A - AC)$: and this is the method of solution adopted also by Mr. Simpson. Mr. Morgan, however, in his hurry to attack M. De Moivre's hypothesis, has inadvertently called this formula an *absurd* one; and says that the error arises from Mr. Simpson's having been misled by that *hypothesis* in determining the probability of one life dying after the other: see *Phil. Trans.* vol. lxxxi, page 276. The present investigation, from the *real probabilities of life*, will show that accusation to be unfounded. Mr. Morgan has wholly mistated the case, as he will readily perceive on re-perusing what he has there written: he has confounded Mr. Simpson's 34th and 36th problems together; and thereby brought an unmerited censure upon that author. The error does not arise from the use of De Moivre's hypothesis, (as he would wish us to believe) but from the inaccurate method which he, in common with Mr. Morgan, has adopted in order to express the chance of one life dying before or after the other during the probable time of their joint continuance: and the same *absurd* formula (if I may retort the self-confuting charge) will equally arise, as I have above observed, even according to Mr. Morgan's own method of solution.

three lives, it will be equal to $x(A - AC) + (\frac{1}{2} - x) \times (AB - ABC) + (1 - \phi - x) \times (A)^d + x(AC)^d$. When C is the oldest of the three lives, it will be equal to $x(A - AC) \times (\frac{1}{2} - x) \times (AB - ABC) + (1 - \phi - x) \times (A)^d - (1 - x) \times (AB)^d$. When the two lives in possession are of the same age B, it will be equal to $\frac{1}{2}(A - AB)^*$. And when all the lives are of the same age A, it will be equal to $\frac{1}{2}(A - AA)^\dagger$.

GENERAL SCHOLIUM.

§ 172. It now remains only to determine the value of x in order to obtain the proper solution of these three problems: and, if the chance of one life dying before or after another in every year of their joint existence were in a constant ratio, we should find no difficulty therein. But since (in computing from the real probabilities of life) this chance is continually varying, we must have recourse to an approximation towards the mean value of such ratio.

§ 173. Now I have found, from a number of re-

* This formula will not differ much from the true value, when the two lives in possession are *nearly* of the same age.

† Many other cases of survivorship might be produced, which involve the contingencies mentioned in Problems XIX and XX and which are solved by the help of those problems in the manner here stated. But, after this investigation, I presume the reader will not find any difficulty in the solution of any other questions of this kind that may occur in practice.

peated trials, that the value of χ may, when $\#B$ is the youngest of the two lives B and C, be safely expressed by $\frac{(1-\phi)b-\beta}{b-\beta}$; and that when C is the youngest of the two lives B and C it may be safely expressed by $\frac{(1-\phi)c}{c-\gamma}$: which, though not in all cases strictly correct,* will come nearer to express the true value of the reversionary annuity than by making χ indiscriminately equal to $\frac{1}{2}$, whatever be the ages of B and C; and may be used till its true value be more correctly determined. But, should a more accurate expression for the value of χ be hereafter found, the general solution of the three problems will not be at all affected thereby, since we may give to χ all possible values.

I shall now insert a few examples, in order to show the use and application of these several formulæ.

§ 174. *Example 1.* What is the value of an annuity on the life of A aged 60 after B aged 40, provided B dies after another life C aged 20: interest being reckoned at 4 per cent, and the probabilities of living as at *Northampton*?

Here we shall have $A=9.039$, $AB=7.490$, $AC=7.995$, $ABC=6.722$, and $\chi=.395$: consequently the value of the reversionary annuity required will in this case be equal to $.276 \times .395 = .109$. But, if B

* It cannot be strictly correct, because the true value of ϕ cannot be deduced by the method pursued in the lemmata; as I have already observed in the note in page 109.

had been 20 and C 40 years of age, we should have $x = .605$: consequently the value of the reversionary annuity would in this case be equal to $.276 \times .605 = .167$.

In like manner, if A had been 20, B 60, and C 40 years of age, we should have $A = 16.033$, $AB = 7.995$, $AC = 10.924$, $ABC = 6.722$, $n = 37$, $x = .354$, and $\phi = .7118$: consequently the value of the reversionary annuity would in this case be equal to $3.836 \times .354 - .067 + .026 = 1.317$. But, if B had been 40 and C 60 years of age, the value would in such case come out equal to 2.519.*

§ 175. *Example 2.* What is the value of an annuity on the life of A aged 60 after B aged 40 provided B dies before another life C aged 20: interest at 4 per cent, and the probabilities of living at Northampton?

* It may here be useful to remark that, if the value of the annuity on the life A after the longest of the two lives B and C (provided B dies after C) be once found, we may readily determine the value of the same annuity, on the contingency that C dies after B, by subtracting the value, thus found, from the whole value of the reversionary annuity on the life A after the longer of the two lives B and C. Thus, the value found by the example given in the text being equal to 1.317, and the latter value here alluded to being equal to 3.836, it follows that $3.836 - 1.317 = 2.519$ will be the value required.

In order that the reader may see the difference in the results according to Mr. Morgan's formulæ and those which are here given, I shall insert the following comparative values of annuities

Here we shall have $\bar{A} - AB = 9.039 - 7.490 = 1.549$: consequently $1.549 - .109 = 1.440$ will (agreeably to what has been said in the scholium in page 135) be the value of the reversionary annuity in this case required. But if A had been 20 and C 60 years of age the value of the reversionary would in such case be equal to $5.109 - 2.519 = 2.590$.

on the life A after B, provided B dies after C, deduced from the Northampton tables, and reckoning interest at 4 per cent: which difference arises from the inaccurate method, adopted by Mr. Morgan, of taking it as an *equal* chance in all cases that B will die after C, whatever be their difference of age.

Age of A.	Age of B.	Age of C.	Value by Baily.	Value by Morgan.
60	40	20	.109	.138
60	20	40	.167	.138
20	60	40	1.317	1.717
20	40	60	2.519	2.120
40	60	20	.519	.716
40	20	60	.986	.789

CHAPTER VI.

On Assurances.

§ 176. IN the preceding chapters I have considered the present value of sums of money as depending on the existence of any given lives, or on any particular survivorship between them; and, in the solution of the different problems relative thereto, have had regard only to the probability of the *living* of those persons on whom the annuity was considered as depending. I come now, however, to treat of those cases where it is required to find the value of annuities, or of sums of money, depending on the *extinction* of any lives; or, in other words, to treat of the value of *Assurances on lives*.* a term applied to that com-

* The term *Assurance* is usually applied only to the value of annuities or sums of money to be paid *after the extinction* of any given lives: but, it may, with equal propriety, be applied to the value of those annuities which are paid *during the existence* of any given lives; and which have been the subject of the preceding chapters. For, if I give a sum of money for the grant of an annuity during the continuance of any given lives, I give such sum in order to have the annuity *assured* to me; which, without this warrantry would be precarious and uncertain. As I am ignorant of any other word, but such as would be equally ambiguous and indefinite, I have used the term *assurance*, in its most common acceptance, to express the values treated of in the present chapter.

fact whereby security is granted for the payment of an annuity or sum of money on the expiration of the lives on which the grant is made, in consideration of such a previous payment made to the assurer, as is accounted a sufficient compensation for the chance of loss to which he exposes himself. The value of this payment (commonly called the *Premium*), in all the principal cases which arise out of this subject, it is my object in the present chapter to determine.

§ 177. It may here, however, be necessary previously to observe that the method to be pursued, for determining the value of any *sum* depending on the extinction of any given lives, will be materially different from that which is pursued for determining the value of any *annuity* under the same circumstances. In the latter case, the expectant is to receive *several* yearly rents, the expectation of receiving *each* of which is independent of his expectation of receiving any other of them. But, in reversionary *sums* the case is very different: for, here, only *one gross sum* is to be received at the extinction of the given lives; and therefore the expectation of receiving it at the end of any *one* year will depend on its *not* having been received in any preceding year: or, which is the same thing, the chance of receiving the sum at the end of any year will be compounded of the probability of the *given* lives failing *in* that year, and of their having continued through all the preceding years. This,

however, will more fully appear in the following investigations.

PROBLEM XXII.*

§ 178. To determine the present value of a **given** sum payable at the end of the year in which **any** number of lives become extinct.

SOLUTION.

Let us in the present investigation confine the **case** to three joint lives A B C, whose probabilities of continuing 1, 2, 3, &c, years are as expressed in § 24: and let the given sum be denoted by s . Now, the present value of such sum, certain to be received at the end of one year, is equal to $s(1+p)^{-1}$: but, as the chance of receiving this sum, at the end of the first year, depends on the joint lives failing in that year, the probability of which is $\frac{abc - a'b'c'}{abc}$, we must multiply the present value of the sum above mentioned by this probability; which will give $s(1+p)^{-1} \times \frac{abc - a'b'c'}{abc}$ for

* Simpson, Prob. 21, and *Sup.* Prob. 26. De Moivre, Prob. 16. Dodson, vol. ii. Ques. 89. Morgan, Prob. 8. Price, Note (E). On referring, however, to the first three authors, here alluded to, it will be found that their investigations are erroneous; in as much as they consider a given *sum* as equivalent to the perpetuity of an *annuity* (commencing immediately) equal to the interest of such *sum*. See the Scholium in § 197.

the true value of the first year's expectation, or of the chance of receiving such sum at the end of the first year.

In like manner, the present value of the given sum, certain to be received at the end of two years, is equal to $s(1+\rho)^{-2}$: but, as the chance of receiving such sum at the end of the second year depends on the joint lives failing in that year, the probability of which (by § 27) is $\frac{a'b'c' - a''b''c''}{abc}$, we must multiply the present value of the sum above mentioned by this probability; which will give $s(1+\rho)^{-2} \times \frac{a'b'c' - a''b''c''}{abc}$ for the true value of the expectation of receiving such sum at the end of the second year.

And by a similar method of proceeding it will be found that, since the chance of receiving the given sum at the end of the third year depends on the joint lives failing in that year, the probability of which is $\frac{a''b''c'' - a'''b'''c'''}{a'b'c'}$, we shall have $s(1+\rho)^{-3} \times \frac{a''b''c'' - a'''b'''c'''}{a'b'c'}$ for the true value of the expectation of receiving such sum at the end of the third year. And so on for every subsequent year of human life: the sum of all which values will be the required present value of the given sum.

§ 179. But the sum of these quantities, reduced to their simplest terms, is equal to the two following

series; $\frac{s}{abc(1+\rho)} \left[abc + \frac{a'b'c'}{(1+\rho)} + \frac{a''b''c''}{(1+\rho)^2} + \frac{a'''b'''c'''}{(1+\rho)^3} + \&c \right]$

$$- \frac{s}{abc} \left[\frac{a^1 b^1 c^1}{(1+\rho)} + \frac{a^{''} b^{''} c^{''}}{(1+\rho)^2} + \frac{a^{'''} b^{'''} c^{'''}}{(1+\rho)^3} + \&c \right]$$
 the former of which is equal to $\frac{s}{(1+\rho)}(1+ABC)$; and the latter, which is to be subtracted, is equal to $s \times ABC$. Consequently the total present value required becomes $s \left[\frac{1+ABC}{(1+\rho)} - ABC \right] = s \times \frac{1-\rho ABC}{(1+\rho)}$: and though this case is confined to three lives, yet it is easy to see that the method of solution is general, and will apply to any other number of lives; whence the following rule.

§ 180. *Multiply the value of an annuity on the given lives by the rate of interest, and subtract the product from unity; divide the remainder by the amount of £1 in one year, and the quotient multiplied by the given sum will be the value required.*

For examples of the use and application of this problem, see Question XXVII in Chapter XII.

COROLLARY I.

§ 181. In this problem I have considered the present value of the reversionary sum as depending on the extinction of the given lives in *whatever* year that may happen during the probability of their joint continuance: but, if this contingency is *Deferred* for any number of years ($=n$); that is, if we wish to ascertain the present value of a given sum payable on the failure of such lives, provided that shall happen *after*

the given period, the formula will be materially altered. For, by pursuing the same steps as in the problem, it will be found that the expectation of receiving the given sum, at the end of the $(n+1)^{\text{st}}$, $(n+2)^{\text{nd}}$, $(n+3)^{\text{rd}}$, &c. years, to the utmost extent of human

life, will be denoted by the series $s \times \left[\frac{a\beta\gamma - a'\beta'\gamma'}{abc(1+\rho)^{n+1}} + \frac{a'\beta'\gamma' - a''\beta''\gamma''}{abc(1+\rho)^{n+2}} + \frac{a''\beta''\gamma'' - a'''\beta'''\gamma'''}{abc(1+\rho)^{n+3}} + \&c \right]$: which may be more conveniently divided into the two following series,

$$\frac{s}{abc(1+\rho)^n} \times \left[\frac{a\beta\gamma}{(1+\rho)} + \frac{a'\beta'\gamma'}{(1+\rho)^2} + \frac{a''\beta''\gamma''}{(1+\rho)^3} + \&c \right] -$$

$$\frac{s}{abc(1+\rho)^n} \times \left[\frac{a'\beta'\gamma'}{(1+\rho)} + \frac{a''\beta''\gamma''}{(1+\rho)^2} + \frac{a'''\beta'''\gamma'''}{(1+\rho)^3} + \&c \right].$$

The latter of these is, by Prob. I, cor. 3, equal to $-s(ABC)^d$; and the former is equal to $s \times \frac{\frac{a\beta\gamma}{abc}(1+\rho)^{-n} + (ABC)^d}{(1+\rho)}$: consequently the required present value of the deferred assurance will be equal to $s \times \frac{\frac{a\beta\gamma}{abc}(1+\rho)^{-n} - \rho(ABC)^d}{(1+\rho)}$; and whence the following rule.*

* Since $(ABC)^d$ is, by Prob. I, cor. 3, equal to $\frac{a\beta\gamma}{abc} \times \frac{1-\rho(ABC)^d}{(1+\rho)}$; it is obvious that the present value, in the case of single or joint lives, might be more conveniently expressed by $s \times \frac{1-\rho(ABC)^d}{(1+\rho)} \times \frac{a\beta\gamma}{abc}(1+\rho)^{-n}$: and it is from this formula that I have deduced the rule in Question XXVIII, Chapter XII. But that rule will not extend to all cases.

§ 182. Multiply the value of a Deferred annuity on the given lives, by the rate of interest; subtract the product from the expectation that the given lives shall receive £1 at the end of the given term; and divide the difference by the amount of £1 in a year: the quotient thence arising, being multiplied by the given sum, will produce the value required.

For examples of the use and application of this corollary, see Question XXVIII in Chapter XII.

COROLLARY II.

§ 183. But, if the contingency is *Temporary*; that is, if we wish to ascertain the value of a given sum payable on the failure of such lives, provided that shall happen within a given term ($=n$), it will be found by taking the first n terms of the series given in the problem. Whence, such present value will in

this case be denoted by the series $s \times \left[\frac{abc - a'b'c'}{abc(1+\rho)} + \frac{a'b'c' - a''b''c''}{abc(1+\rho)^2} + \frac{a''b''c'' - a'''b'''c'''}{abc(1+\rho)^3} + \dots + \frac{a^{(n-1)}b^{(n-1)}c^{(n-1)} - a^{(n)}b^{(n)}c^{(n)}}{abc(1+\rho)^n} \right]$;

which may be more conveniently divided into the following ones; $\frac{s}{abc(1+\rho)} \left[abc + \frac{a'b'c'}{(1+\rho)} + \frac{a''b''c''}{(1+\rho)^2} + \frac{a'''b'''c'''}{(1+\rho)^3} + \dots + \frac{a^{(n-1)}b^{(n-1)}c^{(n-1)}}{(1+\rho)^{n-1}} \right] - \frac{s}{abc} \left[\frac{a'b'c'}{(1+\rho)} + \frac{a''b''c''}{(1+\rho)^2} + \frac{a'''b'''c'''}{(1+\rho)^3} + \dots + \frac{a^{(n)}b^{(n)}c^{(n)}}{(1+\rho)^n} \right]$. But, the

first of these is equal to $s \times \frac{1+(ABC)^{t-1}}{(1+\rho)}$;* and the second, which is to be subtracted, is evidently equal to $s(ABC)^t$: consequently the required value of the assurance for the given term might be denoted by the formula $s \left[\frac{1+(ABC)^{t-1}}{(1+\rho)} - (ABC)^t \right]$.

But, since $(ABC)^{t-1} + \frac{\alpha\beta\gamma}{abc(1+\rho)^n}$ is equal to $(ABC)^t$,† whereby $(ABC)^{t-1}$ becomes equal to $(ABC)^t - \frac{\alpha\beta\gamma}{abc} \times (1+\rho)^{-n}$; and since, by Prob. I, cor. 4, $(ABC)^t$ is equal to $ABC - (ABC)^d$; we may render the above formula more convenient for general use by substituting such value therein instead of $(ABC)^t$: whereby it will become equal to $s \times \frac{1-\rho ABC - \left[\frac{\alpha\beta\gamma}{abc} (1+\rho)^{-n} - \rho (ABC)^d \right]}{(1+\rho)}$; and whence the following rule,

§ 184. *From the present value of the assurance of the given sum payable on the extinction of the*

* The new character $(ABC)^{t-1}$ denotes the value of a temporary annuity on the joint lives ABC for $n-1$ years only, and is introduced here merely to show the steps of the process; and because the rules, given by Mr. Morgan, for the solution of this and other similar questions, are founded on one of these incomplete cases. See Price's *Obs. on Rev. Pay.* vol. i. p. 53, note x; and 1, note e.

As is evident by continuing the series which this quantity expresses.

given lives, subtract the present value of the same as—
 insurance Deferred for the given term: the difference—
 will be the value required.*

For examples of the use and application of this co—
 rollary, see Question XXIX in Chapter XII.

* We may oftentimes obtain a *near* value of an assurance for
 give term (particularly when only one life, A, is concerned) by
 the following simple method. The series expressing such value is,
 by the corollary, equal to $\frac{s}{a} \times \left[\frac{a-a'}{(1+\rho)} + \frac{a-a''}{(1+\rho)^2} + \frac{a-a'''}{(1+\rho)^3} + \dots + \frac{a-a^{(n)}}{(1+\rho)^n} \right]$; now, if we suppose the decreme—
 of life to be equal and uniform throughout the given term (wh^hich
 is usually the case in the middle ages of life, particularly if the
 Northampton tables are made use of), and if such decrements
 be denoted by δ ; then will this series become $\frac{s}{a} \left[\frac{\delta}{(1+\rho)} + \frac{\delta}{(1+\rho)^2} + \frac{\delta}{(1+\rho)^3} + \dots + \frac{\delta}{(1+\rho)^n} \right] = \frac{s \cdot \delta}{a} \times$
 $\frac{1-(1+\rho)^{-n}}{\rho}$. But, since δ should in all those cases where the
 decrements are not *exactly* uniform be taken equal to $\frac{a-a}{n}$, this
 formula may be more conveniently expressed by $s \times \frac{a-a}{an} \times$
 $\frac{1-(1+\rho)^{-n}}{\rho}$: whence the following rule.

Divide the probability that the life shall fail in the given term by
 the number of years; multiply the quotient by the present value—
 of an annuity for the given term, and also by the given sum: the p—
 roduct will be the value required.

It will readily appear that, if the given term is wholly with—
 in the limits of equal decrements, the above formula will give the
 exact value of the assurance for such term: and that if the decr—

COROLLARY III.*

§ 185. Though in this problem the investigation is confined to the case of three joint lives, yet it is easy to apply the same method of reasoning to the case of any single life, or to any number of joint lives, or to the longest of any number of lives, or to any number of lives out of any other number of lives, &c. And it will be found that by substituting the value of an annuity on such single life, on such joint lives, on such longest lives, &c, &c, instead of the quantity denoted by ABC in the given formula in § 179, we shall obtain the true values in such cases accordingly.

§ 186. The same remark will also apply to the two preceding corollaries, if (in addition to the substitution above alluded to) we also substitute the value of similar annuities Deferred for the given term, instead of the quantity $(ABC)^d$ in either formula in

ments are not exactly uniform, but nearly so, it will not materially differ from the true value of such assurance. The formula, however, must always be used with proper caution, and with reference to the table of observations employed; as it is not suited to every case.

The same observations will apply to the case of joint lives, if δ be taken equal to $\frac{abc - a\beta\gamma}{n}$: but we are not so likely, in practice, to meet with cases of assurances on joint lives for terms, where such terms fall within the limits of equal decrements.

* Simpson, Prob. 22, and Sup. Prob. 27. Morgan, Prob. 8, Cor. and Prob. 9, cor.

§ 181 or 183. But here it should be particularly observed that, when a Deferred or Temporary assurance depends on the *longest* of any number of lives, the quantity $\frac{a\beta\gamma}{abc}(1+\rho)^{-n}$ in the given formula will denote the expectation that the *longest* of such lives will receive £1 at the end of the given term: and that, when such assurance depends on the extinction of any *two out of three* lives, the same quantity will denote the expectation that any *two out of three* lives will receive £1 at the end of the given term: &c. &c.

For examples of the use and application of this corollary, see Questions XXVII, XXVIII and XXIX in Chapter XII.

PROBLEM XXIII.*

§ 187. To find the value of an *annuity* to commence at the end of the year† in which any number of lives become extinct.

SOLUTION.

Let us, as in the preceding problem, confine the case to three joint lives ABC, whose probabilities of continuing 1, 2, 3, &c, years are as denoted in § 24: and let the annuity be a perpetuity, or estate in fee. Now, the chance which the heir of this estate has of receiving the rent at the end of any one year will de-

* De Moivre, Prob. 6 and 29. Dodson, vol. ii. Ques. 88.

† That is, the *first* payment of the annuity is to be then made.

pend altogether on the joint lives becoming extinct before the end of that year. The probability of this event happening before the end of the first year is $(1 - \frac{a'b'c}{abc})$, which being multiplied by $(1 + \rho)^{-1}$ will give the value of his expectation of receiving the first year's rent. In like manner, the probability of the joint lives failing before the end of the second year is $(1 - \frac{a''b''c''}{abc})$, which being multiplied by $(1 + \rho)^{-2}$ will give the value of his expectation of receiving the second year's rent. By a similar method of reasoning it will be found that $(1 - \frac{a'''b'''c'''}{abc})$ multiplied by $(1 + \rho)^{-3}$ will give the value of his expectation of receiving the third year's rent. And so on, *ad infinitum*: for, since the estate must eventually revert to the heir, his expectation will never cease. Consequently the sum of all those values continued to infinity will be the total present value of the estate to be enjoyed after the extinction of the joint lives ABC: or, in other words, it will be that sum which a person ought to give to have it assured to him after the extinction of those lives.

§ 188. But the sum of the above terms is equal to the two following series; $\frac{1}{(1 + \rho)} + \frac{1}{(1 + \rho)^2} + \frac{1}{(1 + \rho)^3} + \dots \dots \dots \text{ad infinitum} - \frac{1}{abc} \times \left[\frac{a'b'c}{(1 + \rho)} + \frac{a''b''c''}{(1 + \rho)^2} + \frac{a'''b'''c'''}{(1 + \rho)^3} + \&c \right]$. The former of these is equal to $\frac{1}{\rho}$, or the present value of the perpetuity of an annuity of

£1 per annum: but the latter, which is to be subtracted, will cease on the extinction of the oldest life, and is therefore equal to the value of an annuity on the joint lives ABC. Consequently the total value of the reversionary annuity becomes equal to $\frac{1}{p} - ABC$: and though this case is confined to that of three joint lives, yet it is easy to see that the method of solution will equally apply to that of any other number of lives; whence the following rule.

§ 189. *Subtract the value of an annuity on the given lives from the present value of the estate or given annuity; the difference will be the value of the reversionary estate or annuity required.*

For examples of the use and application of this problem, see Question XXVI in Chapter XII.

COROLLARY I.

§ 190. If the annuity, instead of being a perpetuity, is only for a number of years ($=n$) greater however than that to which it is probable the given lives may extend,* we must substitute the value of such termi-

* The term to which it is probable that any given life or lives may extend is—for a *single* life, equal to the difference between the age of such life and the age of the oldest life in the table of observations:—for *joint* lives, equal to the difference between the *oldest* of such lives and the age of the oldest life in the table:—for the *longest* of any lives, equal to the difference between the age of the *youngest* of such lives and the age of the oldest life in the table.

nable annuity instead of the value of the perpetuity above mentioned: whereby the formula becomes $\frac{1-(1+\rho)^{-n}}{\rho} = ABC$; and consequently the rule above given is also applicable to the present case.

For examples of the use and application of this corollary, see Question XXVI in Chapter XII,

COROLLARY II.*

§ 191. But, if the annuity (instead of being a perpetuity, or for any long period) is for a term of years ($=n$) which is less than that to which it is probable the given lives may extend; the series, expressing the required value in the problem, will terminate at the end of that period; and will consequently become equal to $\frac{1}{(1+\rho)} + \frac{1}{(1+\rho)^2} + \frac{1}{(1+\rho)^3} + \dots$
 $\frac{1}{(1+\rho)^n} - \frac{1}{abc} \left[\frac{abc}{(1+\rho)} + \frac{abc}{(1+\rho)^2} + \frac{abc}{(1+\rho)^3} + \dots \right]$. But, the first part is equal to $\frac{1-(1+\rho)^{-n}}{\rho}$; and the latter part is, by Prob. I, cor. 4, equal to $-(ABC)^{\dagger}$: whence the following rule.

§ 192. *From the value of an annuity certain for the given term, subtract the value of a Temporary annuity on the given lives for the given term: the difference will be the value required.*

* Simpson's *Sup. Prob.* 14. De Moivre, *Prob.* 28. Morgan, *Prob.* 4, cor. 2. Dodson, vol. ii. *Ques.* 92.

COROLLARY III.

§ 193. If the contingency (on which the payment of the annuity mentioned in the problem depends) is *Deferred* for a given number of years ($=n$); the series expressing the general value of the reversion of the perpetuity must commence at the end of that period, and be continued on to infinity; that is, the

series $\frac{1 - \frac{a' \beta' \gamma'}{a \beta \gamma}}{(1+\rho)^{n+1}} + \frac{1 - \frac{a'' \beta'' \gamma''}{a \beta \gamma}}{(1+\rho)^{n+2}} + \frac{1 - \frac{a''' \beta''' \gamma'''}{a \beta \gamma}}{(1+\rho)^{n+3}} + \dots$ *ad infinitum*, will denote the value of the required reversionary annuity, to be entered on provided either of the given lives become extinct after the n^{th} year. Now this series may be resolved into two others, namely, $(1+\rho)^{-n} \times \left[\frac{1}{(1+\rho)^1} + \frac{1}{(1+\rho)^2} + \frac{1}{(1+\rho)^3} + \dots \right]$ *ad infinitum* $]- \frac{1}{a \beta \gamma (1+\rho)^n} \left[\frac{a' \beta' \gamma'}{(1+\rho)^1} + \frac{a'' \beta'' \gamma''}{(1+\rho)^2} + \frac{a''' \beta''' \gamma'''}{(1+\rho)^3} + \dots \right]$: the first of which is equal to $(1+\rho)^{-n} \times \frac{1}{\rho}$; and the latter is, by Prob. I, cor. 3, equal to $-(1+\rho)^{-n} \times ABC'$. Consequently, the value of the reversionary annuity would be equal to $(1+\rho)^{-n} \times \left[\frac{1}{\rho} - ABC' \right]$ were it certain that the given lives would continue in being to the end of n years: but, as the probability of this event is $\frac{a \beta \gamma}{abc}$, we must multiply the above expression by this quantity, in order to obtain the total present value of

the same. Whence the true value will be equal to $\frac{a\beta\gamma}{abc}(1+\rho)^{-n} \times \frac{1}{\rho} - A'B'C \times \frac{a\beta\gamma}{abc}(1+\rho)^{-n}$. But $A'B'C \times \frac{a\beta\gamma}{abc}(1+\rho)^{-n}$ is, by Prob. I, cor. 3, equal to $(ABC)^d$; therefore the above value becomes $\frac{a\beta\gamma}{abc}(1+\rho)^{-n} \times \frac{1}{\rho} - (ABC)^d$: and whence the following rule.*

§ 194. *Multiply the expectation of the given lives receiving £1 at the end of the given term by the perpetuity; from the product subtract the value of an annuity on the given lives Deferred for the given term: the difference will be the value required.*

COROLLARY IV.†

§ 195. Or, if the contingency is *Temporary*; that is, if it be required to ascertain the value of such annuities depending on the extinction of any given lives, provided that shall happen *within a given period* ($= n$); the value of the assurance will be expressed

By the series $\frac{1 - \frac{a' b' c'}{a b c}}{(1-\rho)} + \frac{1 - \frac{a'' b'' c''}{a b c}}{(1+\rho)^2} + \frac{1 - \frac{a''' b''' c'''}{a b c}}{(1+\rho)^3} + \dots$
 $1 - \frac{a\beta\gamma}{abc} + \frac{1 - \frac{a\beta\gamma}{abc}}{(1+\rho)^{n+1}} + \frac{1 - \frac{a\beta\gamma}{abc}}{(1+\rho)^{n+2}} + \text{ad infinitum.}$ For,
 the n^{th} year's rent and all the subsequent ones being

* See the note in page 147.

† Simpson, Prob. 23; and *Sup.* Prob. 31. Price, Note (G). Dodson, vol. ii. Ques. 93.

dependent on the given lives becoming extinct in n years, it is obvious that all such subsequent ones must be multiplied by the common factor $(1 - \frac{\alpha\beta\gamma}{abc})$. But, the first n terms of this series are equal to $\frac{1 - (1+\rho)^{-n}}{\rho} - (ABC)^t$; as found by the second corollary: and the remaining terms are equal to $(1 - \frac{\alpha\beta\gamma}{abc}) \times (1 + \rho)^{-n} \times [(1 + \rho)^{-1} + (1 + \rho)^{-2} + (1 + \rho)^{-3} + \dots \dots \dots \text{ad infinitum}] = (1 - \frac{\alpha\beta\gamma}{abc}) \times (1 + \rho)^{-n} \times \frac{1}{\rho}$. Consequently the total value of the series becomes $\frac{1 - (1+\rho)^{-n}}{\rho} - (ABC)^t + \frac{(1+\rho)^{-n}}{\rho} - \frac{\alpha\beta\gamma}{abc} (1 + \rho)^{-n} \times \frac{1}{\rho}$: which, since $(ABC)^t = ABC - (ABC)^d$, may be further reduced to $\frac{1}{\rho} - ABC - [\frac{\alpha\beta\gamma}{abc} (1 + \rho)^{-n} \times \frac{1}{\rho} - (ABC)^d]$: and whence the following rule.*

* If the annuity is to be entered upon *at the end of the given time* (in case of the failure of the given lives), and not *at the end of the year in which such failure may happen*, its present value will be equal to $(1 - \frac{\alpha\beta\gamma}{abc}) \times (1 + \rho)^{-n} \times (1 + \frac{1}{\rho}) = (1 - \frac{\alpha\beta\gamma}{abc}) \frac{(1 + \rho)^{-(n-1)}}{\rho}$: for, such reversionary perpetuity is evidently to be entered on and enjoyed at the end of the n^{th} year, provided the given lives fail prior thereto. And care should be taken not to confound this case with the one mentioned in the text.

§ 196. *From the whole value of the reversion of the perpetuity after the given lives, subtract the value of the same Deferred for the given term: the difference will be the value required.*

COROLLARY V.

§ 197. It will be obvious, from what has been said in the first corollary, that if the annuity, instead of being a perpetuity, is for a given term only, the rules given in the two preceding corollaries will still be correct; provided we substitute, for the *perpetuity*, the value of an annuity for the *given term*. Such terminable annuity however must always be for a term of years *greater* than that, for which the assurance is made.

COROLLARY VI.

§ 198. Though the above problem and its corollaries are confined, in the investigation, to the case of three joint lives, yet it is easy to apply the same method of reasoning to the case of any single life, or to any number of joint lives, or to the longest of any number of lives,* or to any number of lives out of any other number of lives, &c, &c, either for the *whole* lives or for a given *term*.† And it will be found that by substituting the value of an annuity on

* Dodson, vol. ii. Ques. 90, 91.

† Simpson, Prob. 24.

such single life, on such joint lives, on the longest of any number of lives, &c, (either for the whole life or Deferred for the given term) instead of the value ABC or $(ABC)^d$ in the given formulæ, we shall obtain the true values in such cases accordingly.

§ 199. It should be particularly observed however that, when the *longest* of any number of lives are concerned, the quantity $\frac{a\beta\gamma}{abc}(1 + e)^{-n}$ will represent the expectation that the *longest* of such lives will receive £1 at the end of the given term. And a similar remark will apply when the assurance depends on the extinction of any *two out of three* lives: &c. &c. See Problem XXII, cor. 3.

SCHOLIUM.

§ 200. From these two problems and their corollaries may be determined all questions relative to the value of assurances of any given sum, or of any given annuity, depending on the contingencies therein stated: and as any given sum may be converted into a corresponding perpetuity (or into a perpetual annuity of a corresponding value) by multiplying it by the interest of £1 for a year, it would seem at first to be the same thing to determine the present value of a given *sum* depending on the extinction of any lives, as to determine the value of a corresponding *annuity* for ever, depending on the same lives: that is, it would appear to be the same thing to determine the present

value of £100 payable at the end of the year in which any given lives became extinct, as to determine the value of an estate yielding £5 per annum, and to be entered upon at the same period; interest being reckoned at 5 per cent. But, it should be observed that the first yearly payment of a reversionary *annuity* becomes due and is payable at the end of the year in which the lives fail; however much or little of that year may then happen to be unexpired: and this likewise is the time when a reversionary *sum* becomes due. The expectant therefore in the former case will have entered on his annuity, or received the first year's rent of it, at the very time that the expectant of the sum is supposed to have laid out such sum in the purchase of a perpetuity of a corresponding value; the first year's rent of which he will not receive till the *end of the following year*. Consequently, a reversionary estate is worth one year's purchase more than a corresponding reversionary sum: whence the former is to the latter in the ratio of £1, increased by its interest for one year, to £1; that is, as $(1+e)$ to 1.

§ 201. Therefore, if the present value of a reversionary sum be multiplied by $(1+e)$ it will give the value of a corresponding reversionary estate or annuity. Thus, the present value of a reversionary annuity of £1 per annum for ever after the extinction of the joint lives ABC is, by Prob. XXIII, equal to $\frac{1-e^{ABC}}{e}$; and the present value of a corre-

sponding reversionary sum ($= \frac{1}{\rho}$) after the same lives, is, by Prob. XXIII, equal to $\frac{1}{\rho} \times \frac{1-\rho ABC}{(1+\rho)} = \frac{1}{(1+\rho)} \times \frac{1-\rho ABC}{\rho}$: but, it is evident on inspection that the former value is to the latter in the ratio of $(1+\rho)$ to 1.

§ 202. Hence it is evident also that the value of any reversionary annuity after any given lives, being divided by the amount of £1 in a year, will give the present value of a corresponding reversionary sum after the same lives.*

These remarks may be more fully confirmed by any of the similar examples in the two problems, or their corollaries: as may be seen in the Scholium to Question XXVII in Chapter XII.

* Mr. Simpson not having attended to this circumstance, it becomes necessary to correct the rules given by him for the solution of Prob. 21 and 22 in his *Doctrine of Annuities, &c.*; and of Prob. 26, 27, 32, 33, and others of a similar kind, in his *Supplement*. The same observation will apply to the problems of M. De Moivre and Mr. Dodson, alluded to in the note in page 144.

CHAPTER VII.

On Successive Life Annuities and Copyhold Estates.

§ 203. IN all the preceding problems, involving the question of reversionary life annuities, the lives of the expectants have been supposed to be such as are *now* fixed on and determined; and the value of an annuity on their lives consequently becomes less and less, according as their period of coming into possession might be prolonged. In such questions, however, as relate to the present division of the subject, the life which is to succeed to the annuity, after the extinction of the life in possession, is supposed to be one which is *then* to be fixed on at pleasure; and which will probably be one of the best lives that can then be found. This life, therefore, may be considered as having a fixed and determinate value, since such a life may generally be chosen as will best answer the views of persons concerned in questions of this kind; and it is usual to conceive a *mean* age at which they are all admitted. But, the nature of the cases, which involve the consideration of this subject, will best appear from the following problems.

PROBLEM XXIV.*

§ 204. Supposing *A* to enjoy an annuity for his life; and, at his decease, to have the nomination of a successor, *B*, who is also to enjoy the annuity for his life: To find the present value of the annuity on the succeeding life; and also the value of the two successive lives.

SOLUTION.

Let the succeeding life *B*, to be put in nomination at the decease of *A*, be such that an annuity on his life *at that time* may be equal to *B*. Now, since the probability that the first life fails, or that the second comes into possession, in the first year is $\frac{a-a'}{a}$; and as the total value of what the second life will be entitled to, on the happening of this event, is *B*, it follows that his expectation of coming into possession the first year (or of receiving an equivalent sum, equal to *B*) will, as in Prob. XXII, be denoted by $B \times \frac{a-a'}{a(1+\rho)}$. In like manner, it will be found that his expectation of coming into possession in the second year (or of receiving an equivalent sum *B*) will be denoted by $B \times \frac{a-a''}{a(1+\rho)^2}$: and that his expectation of coming into

* Simpson, Prob. 25, and *Sup.* Prob. 24. De Moivre, Prob. 13. Morgan, Prob. 14.

possession in the third year (or of receiving the equivalent sum B) will be denoted by $B \times \frac{\frac{a'' - a'''}{a(1+\rho)^3}}$: and so on for every succeeding year to the utmost extent of A 's life.

But the sum of all these values is equal to $\frac{B}{a(1+\rho)} \times \left[a + \frac{a'}{(1+\rho)} + \frac{a''}{(1+\rho)^2} + \frac{a'''}{(1+\rho)^3} + \&c \right] - \frac{B}{a} \times \left[\frac{a'}{(1+\rho)} + \frac{a''}{(1+\rho)^2} + \frac{a'''}{(1+\rho)^3} + \&c \right]$; which, by Prob. XXII, is equal to $B \times \frac{1-\rho A}{(1+\rho)}$; and which would be the value required, were B in reality the value of a reversionary sum to be received on the decease of A : but since it denotes the value of an annuity, the first payment of which commences at the end of the year in which the life A fails, we must multiply the above expression by $(1+\rho)$, agreeably to what has been said in the scholium in page 160; whence, the true present value of the successive life will be $B(1-\rho A)$; and whence the following rule.

§ 205. *Multiply the value of an annuity on the life in possession by the rate of interest, and subtract the product from unity; multiply the remainder by the assumed value of an annuity on the succeeding life: the product will be the present value of an annuity on such succeeding life.*

§ 206. *If this present value be added to the value of an annuity on the life in possession, it will give*

$A + B(1 - p^A)$ for the value of the *two successive* lives.

For examples of the use and application of this problem see Question XXIII in Chapter XII.

COROLLARY I.

§ 207. Hence may be determined the present value of an annuity on any number of *joint* lives, or on the *longest* of any number of lives, &c, nominated to succeed after any other number of joint lives, or after the longest of any other number of lives, &c: for, by making A and B , respectively, equal to the value of an annuity on such joint lives, or the longest of such lives, &c, the above formula will express the true present values of such successive lives.

COROLLARY II.*

§ 208. If the succeeding life, instead of receiving an annuity *during his life*, were to receive an annuity *certain for a given term of years* after the failure of the life in possession; then, by making B equal to the value of such annuity certain for the given term, the above formula would truly express the present value of such annuity to be entered on at the failure of the life in possession.

And if this annuity were a perpetuity (that is, if the succeeding life and his heirs were to receive

* Morgan, Prob. 13.

annuity for ever after the failure of the life in possession) B would become equal to $\frac{1}{\rho}$, and the formula would in this case become equal to $\frac{1}{\rho} - A$; which is the very same as that deduced from Prob. XXIII.

For examples of the use and application of this corollary see Question XXIV in Chapter XII.

PROBLEM XXV.*

§ 209. Three lives A, B, C , being given in succession: To find the present value of an annuity on the third succeeding life: and also the value of the three successive lives.

SOLUTION.

Let the values of an annuity on each of the three lives, at the time that they severally come into possession, be respectively denoted by A, B, C . Therefore, since the value of an annuity on the second life in succession (to commence at the decease of A) is to the value of a perpetuity (to commence at the same time) in the ratio of B to $\frac{1}{\rho}$; it follows that the present value of the former will be to the present value of the latter in the same ratio. But, the present value of the

* Simpson, Prob. 26, and *Sup.* Prob. 25.

latter is, by Prob. XXIII, equal to $\frac{1}{\rho} - A = \frac{1-\rho A}{\rho}$; therefore $\frac{1}{\rho} : B :: \frac{1-\rho A}{\rho} : B(1-\rho A)$ = the present value of the former; and which is the same value as that found by the last problem: whence the two methods of solution confirm the truth of each other.

The present value of the first two successive lives being thus found equal to $A + B(1-\rho A)$; it follows that the present value of the reversion of a perpetuity after these lives will (by a deduction from Prob. XXIII) be equal to $\frac{1}{\rho} - [A + B(1-\rho A)] = \frac{1}{\rho}(1-\rho A) \times (1-\rho B)$. Consequently, since the value of an annuity on the third life in succession (to commence at the decease of B) is to the value of a perpetuity (to commence at the same period) in the ratio of $\frac{1}{\rho}$ to $\frac{1}{\rho}$; it follows that the present value of the former will be to the present value of the latter, in the same ratio: that is, $\frac{1}{\rho} : C :: \frac{1}{\rho}(1-\rho A) \times (1-\rho B) : C(1-\rho A) \times (1-\rho B)$ = the present value of an annuity on the third successive life; whence the following rule.

§ 210. *Multiply the value of an annuity on the first life in possession by the rate of interest, and subtract the product from unity; multiply also the assumed value of an annuity on the second life in succession by the rate of interest, and subtract this product likewise from unity: multiply together these two remainders, and their product again by the assumed value*

of an annuity on the third life in succession: this last product will be the value of the third successive life.

§ 211. If the present value of each successive life, as above found, be added together, their sum, or $A + B(1 - \rho A) + C(1 - \rho A) \times (1 - \rho B)$, will be the present value of the three successive lives: but this expression will be found equal to $\frac{1}{\rho} \left[1 - (1 - \rho A) \times (1 - \rho B) \times (1 - \rho C) \right]$: whence the present value of any number of lives in succession may be discovered on inspection; and thence, the following rule.

§ 212. *Multiply the assumed value of an annuity on each of the proposed lives, by the rate of interest; take the several products from unity, and multiply together all the remainders; let the product thus arising be also subtracted from unity, and the remainder divided by the rate of interest: the quotient will be the present value of all the successive lives, including the life in possession.*

PROBLEM XXVI.*

§ 213. Suppose a person to purchase a copyhold estate, on any number of lives A, B, C, &c, for the sum s , on condition that he and his successors may

* Simpson, Prob. 27; and *Sup.* Prob. 29. Dodson, vol. iii. Ques. 82 to 95. De Moivre, Prob. 10.

renew it continually by paying the fine f whenever any one of such lives becomes extinct: To find the present value of the whole purchase of such estate.

SOLUTION.

Let the value of an annuity on each of the lives A, B, C , &c, in possession, be respectively denoted by A, B, C , &c: and let the value of an annuity on each of the lives A^1, A^2, A^3 , &c, (which are supposed to follow in direct succession from A) be respectively denoted by A^1, A^2, A^3 , &c.* In like manner, let the value of an annuity on each of the lives, B^1, B^2, B^3 , &c, (which are supposed to follow in direct succession from B) be respectively denoted by B^1, B^2, B^3 , &c: and so on with respect to the lives immediately succeeding C, D , &c. And let us first determine the present value of all the fines payable on the extinction of the life A and his immediate successors.

Now, the present value of the fine f payable on the decease of A , in whatever part of the year that may happen, may in the present case be considered equal to the present value of an estate, yielding f per annum, to be entered upon at the decease of A ; which, by Prob. XXIII, is found to be $f \rho \left(\frac{1}{\rho} - A \right) = f(1 - \rho A)$. If instead of A in this formula we sub

* Care must be taken not to mistake these numeral quantities for the *index* of the quantity to which they are annexed: as, they merely denote the order of succession among the given lives.

stitute the present value of the two successive lives $A, \overset{1}{A}$, which, by Prob. XXIV, is found equal to $A + \overset{1}{A}(1 - \rho A)$, we shall have $f(1 - \rho A) \times (1 - \rho \overset{1}{A})$ for the present value of the fine to be paid on the decease of $\overset{1}{A}$. And if instead of A in that same formula we substitute the present value of the three successive lives $A, \overset{1}{A}, \overset{2}{A}$, which, by the last problem, is found equal to $A + \overset{1}{A}(1 - \rho A) + \overset{2}{A}(1 - \rho A) \times (1 - \rho \overset{1}{A})$, we shall have $f(1 - \rho A) \times (1 - \rho \overset{1}{A}) \times (1 - \rho \overset{2}{A})$ for the present value of the fine payable on the decease of $\overset{2}{A}$: and so on with respect to all the subsequent fines payable on the extinction of each life in direct succession from A . But, the sum of all these quantities, or the series $f[(1 - \rho A) + (1 - \rho A) \times (1 - \rho \overset{1}{A}) + (1 - \rho A) \times (1 - \rho \overset{1}{A}) \times (1 - \rho \overset{2}{A}) + \dots \text{ad infinitum}]$ is the present value of all the sums that may be paid from time to time for the renewals of the several lives in direct succession from A .

By a similar method of proceeding it will be found that the series $f[(1 - \rho B) + (1 - \rho B) \times (1 - \rho \overset{1}{B}) + (1 - \rho B) \times (1 - \rho \overset{1}{B}) \times (1 - \rho \overset{2}{B}) + \dots \text{ad infinitum}]$ will denote the present value of all the sums that may be paid from time to time for the renewals of the several lives in direct succession from B . And so on with respect to the other lives C, D , &c, and their successors for ever: and the sum of all these different

$$\begin{aligned}
 & f\left[(1-\rho A)+(1-\rho A)\times(1-\rho A)^1+\dots ad\ inf.\right]+ \\
 & f\left[(1-\rho B)+(1-\rho B)\times(1-\rho B)^1+\dots ad\ inf.\right]+ \\
 & f\left[(1-\rho C)+(1-\rho C)\times(1-\rho C)^1+\dots ad\ inf.\right]+ \\
 & \qquad \qquad \qquad \&c \qquad \qquad \qquad \&c \qquad \qquad \qquad \&c,
 \end{aligned}$$

will be the total present value of all the fines that the tenant can ever pay; and which being added to s will give the whole value paid for the purchase.

COROLLARY I.

§ 214. Hence if the lives, with which the lease is from time to time renewed, be supposed equal to one another, or of the same common age $\dot{A}$, the general expression above given will become

$$\begin{aligned}
 & f(1-\rho A)\left[1+(1-\rho A)^1+(1-\rho A)^2+(1-\rho A)^3+\dots ad\ inf.\right] \\
 & f(1-\rho B)\left[1+(1-\rho A)^1+(1-\rho A)^2+(1-\rho A)^3+\dots ad\ inf.\right] \\
 & f(1-\rho C)\left[1+(1-\rho A)^1+(1-\rho A)^2+(1-\rho A)^3+\dots ad\ inf.\right] \\
 & \qquad \qquad \qquad \&c \qquad \qquad \qquad \&c \qquad \qquad \qquad \&c \qquad \qquad \qquad \&c,
 \end{aligned}$$

the sum of which series, since $\left[1+(1-\rho A)^1+(1-\rho A)^2+(1-\rho A)^3+\dots ad\ infinitum\right]$ is equal to $\frac{1}{1-\rho A}$,* becomes $\frac{f}{A}\left(\frac{1}{\rho}-A+\frac{1}{\rho}-B+\frac{1}{\rho}-C+\&c\right)$

* It is well known that $\frac{1}{1-x} = 1+x+x^2+x^3+\dots ad\ infinitum$: therefore, by substituting $(1-\rho A)$ for x , we shall have $\frac{1}{1-\rho A}$ equal to the series given in the text.

$\frac{f}{A} \left(\frac{n}{\rho} - A - B - C - \&c \right)$; where n denotes the number of lives on which the estate is held: whence following rule.

§ 215. *Divide the number of lives by the rate of interest, and from the quotient subtract the sum of values of an annuity on each of the single lives in session; divide the remainder by the assumed value of an annuity on the common life with which the lease from time to time to be renewed: the quotient, once arising, multiplied by the fine to be paid on renewing will be the total present value of all the renewals for ever: and which, being added to the sum given for the estate, will give the whole value of the purchase.*

§ 216. *Example.* Suppose a person to have paid down £1000 for the purchase of a copyhold estate held on three lives, whose ages are 30, 50 and 70; condition that he may, on the extinction of any, continually renew with any other life that he thinks proper, on paying a fine of £600: What is the present value of all those fines with which the estate may be continually renewed, reckoning interest at 5 per cent, and the probabilities of living according to the observations of *M. De Parcieux*?

Here we shall have $A=14.693$, $B=11.363$, $C=9.55$, $n=3$, $\rho=.05$, $f=600$, and $\dot{A}$ (or the value of

an annuity on the best life in the table*) = 16·213.
 Consequently, $\frac{600}{16\cdot213}(60 - 32\cdot111) = 1402\cdot171$, or
 £1402: 3: 5 will be the present value of all the fines:
 and which being added to the £1000 paid upon en-
 tering will give the total value of the fee simple of the
 estate.

COROLLARY II.

§ 217. When all the lives in possession are of the
 same common age A , the formula in the preceding
 corollary will become equal to $\frac{fn}{A}(\frac{1}{\rho} - A)$. But, if
 all the lives, as well those in possession as those to be
 put in nomination afterwards, be equal to each other,
 or of the same common age A , the present value of
 all the renewals for ever will then be equal to $\frac{fn}{A}$
 $(\frac{1}{\rho} - A) = fn(\frac{1}{\rho A} - 1)$.

COROLLARY III.

§ 218. If the present value found by either of
 the preceding corollaries be multiplied by the rate of
 interest, it will show how much the rent-roll of the
 landlord's estate ought in each case to be increased on
 account of the fines paid at renewing.

* This is always the assumed value of A , agreeably to what
 has been said on this subject in § 203.

Thus, in the example given in § 216, it will be found that $1402.171 \times .05 = 70.109$, or £70 : 2 : 2 is the sum by which the rent-roll of the lord's estate ought to be increased on account of the fines there mentioned.

COROLLARY IV.

§ 219. Since the purchase money paid for the lease, together with the present value of all the fines to be paid on renewal, is equal to the value of the perpetuity of the rack-rent of the estate ($= p$); that is, since $s + \frac{f}{A} \left(\frac{n}{p} - A - B - C - \&c \right) = p$; it follows that f , or the fine which ought to be paid on renewing, will be equal to $\frac{(p-s)A}{\frac{n}{p} - A - B - C - \&c}$: whence the following rule.

§ 220. *Subtract the tenant's interest in the lease (or the purchase money which he has given for the same) from the value of the fee simple of the estate, and multiply the remainder by the assumed value of an annuity on the common life with which the lease is supposed to be constantly renewed; reserving the product:—divide the number of lives on which the lease now held, by the rate of interest; and from the quotient subtract the sum of the values of an annuity each of those lives: the reserved product being divided by this remainder, will give the sum which shall in justice to be paid as a fine on each renewal.*

§ 221. *Example.* Suppose a person to have purchased, for £1000, a copyhold estate, the rack-rent of which is estimated at £100 per annum; and that such estate is held on three lives, renewable for ever on the extinction of either of those lives by paying a fine certain: what ought such fine to be fixed at, in order that the purchaser may make 5 per cent interest of his money, supposing the ages of the lives (on which the estate is now held) to be 30, 50 and 70, and that the probabilities of living are according to the observations of *M. De Parcieux*?

Here we shall have p (or the value of the perpetuity of the rack-rent of the estate) = 2000, $s = 1000$, and the remaining quantities as in the example in § 216. Consequently the value of the fine ought to be $\frac{1000 \times 16.213}{60 - 32.111} = 581.340$, or £581 : 6 : 10.

COROLLARY V.

§ 222. If the estate is held on one life only, the present value of the landlord's interest therein will be universally expressed by $f \times \frac{1 - \rho A}{\rho A}$. Now, immediately after the receipt of a fine, the life in possession is equal to A ; whence the expression in this case becomes $f \times \frac{1 - \rho A}{\rho A}$: and immediately before the receipt of a fine, the life in possession having become extinct, the expression in this case becomes $f \times \frac{1}{\rho A}$.*

* These three formulæ will serve to express the value of perpetual Advowsons (considered as an object of traffic) under

But, since the present value of the next fine is universally expressed by $f(1 - \rho A)$, or by the amount of such fine multiplied into the difference between unity and the product of the rate of interest by the value of an annuity on the life in possession, we may readily determine the landlord's interest in the estate, or the value of all the fines to be paid on renewing, by the following rule.

§ 223. *Divide the present value of the next fine by the product of the rate of interest into the value of an annuity on the common life with which the lease is to be continually renewed: the quotient thence arising will be the value required.*

three most usual circumstances: 1° where the living is *possessed by an incumbent* whose life is equal to A ; 2° immediately *after presentation*, where the life presented is of the same common age, A , as that with which the living is supposed to be constantly filled up; 3° with *immediate resignation*.

It must however be here particularly observed that, in *these cases* the value of A (or the value of an annuity on the common life with which the living is supposed to be constantly filled up) must never be assumed so great as in those cases mentioned in the text: because the person, who is presented to the living, must always be above 24 years of age: and it seldom happens that he is even so young as this.

It has been ingeniously suggested that the ages of the incumbents, when they are inducted, may be partly fixed from the value of the livings. See De Moivre, Prob. 14 and 15: and Dodson, vol. iii. page 347.

§ 224. *Example.* Suppose that a copyhold tenant pays to the lord of the manor a fine of £100 on his admission, and that every successor does the same; what is the present value of the lord's interest in that copyhold, on the supposition that the tenants admitted thereto are (one with another) 25 years of age at the time of their admission: interest being reckoned at 5 per cent, and the probabilities of living as at *Northampton*?

Here we shall have $A = 13.567$, $e = .05$, and $f = 100$; consequently the value of the lord's interest immediately before the receipt of a fine will be equal to $100 \times \frac{1}{.05 \times 13.567} = 147.416$, or £147 : 8 : 4: and immediately after the receipt of a fine it will be equal to $100 \times \frac{1 - .05 \times 13.567}{.05 \times 13.567} = 47.416$ or £47 : 8 : 4. But if the life now in possession be 70 years of age, we shall have $A = 6.023$; in which case, the lord's interest will be $100 \times \frac{1 - .05 \times 6.023}{.05 \times 13.567} = 103.022$, or £103 : 0 : 5.

Therefore if the tenant gave £500 for the lease, the whole value of the purchase may in this latter case be estimated at £603; and the corresponding rent at £30 : 3 : 0.

COROLLARY VI.*

§ 225. If the estate is held on the longest of any

* Simpson's *Sup. Prob.* 28. De Moivre, *Prob.* 12. Dodson, vol. iii. Ques. 85.

number of lives (that is, on condition that, whenever *all* those lives become extinct, the lease may be renewed with the same number of lives, and on the same conditions, by paying the given fine) the formulæ in the last corollary, since all the lives may in this case be considered but as one, will still express the true value of the landlord's interest; if we make A denote the value of an annuity on the *longest* of all the lives in possession, and A^1 the assumed value of an annuity on the *longest* of all the lives with which the lease is to be continually filled up.

§ 226. *Example.* Suppose an estate to be leased on two lives, with condition that, on the extinction of both those lives, the same may be renewed with two other lives (the best that can be found) on paying a fine of £300; and so on for ever: what is the present value of the landlord's interest in the estate: taking the probabilities of living as at *Northampton*, and the rate of interest at 5 per cent.?

Here we shall have A^1 (or the value of an annuity on the longest of two lives, both aged 8 years) = 17.721, $p = .05$, and $f = 300$; consequently the value of the landlord's interest *immediately before* the receipt of a fine will be equal to $300 \times \frac{1}{.05 \times 17.721} = 340.656$, or £340 : 13 : 1: and *immediately after* the receipt of a fine it will be equal to 40.656. But, if the ages of the lives, on which the estate is now held, be 40 and 60 years of age, the landlord's in-

terest will be equal to $300 \times \frac{1 - .05 \times 13.214}{.05 \times 17.721} = 114.880$:
 or, if the eldest of those lives be extinct, the land-
 lord's interest will be equal to $300 \times \frac{1 - .05 \times 11.837}{.05 \times 17.721} =$
 138.192.

SCHOLIUM.

§ 227. From the principles here laid down, it will be easy to determine whether it is most advantageous, to the lessee or the landlord, to fill up a life as soon as it becomes vacant; or to wait till two or more of them have dropt before the renewal.

CHAPTER VIII.

On Assurances depending on a particular Order of Survivorship.

§ 228. THE subject of the present chapter is certainly one of the most intricate in the whole doctrine of annuities, since it involves contingencies for which it is very difficult to give a concise and accurate expression. When two lives only are concerned, the investigations are not very complex; and the solutions may be obtained without much labour or inconvenience: but when three or more lives are involved in the question, the investigations become more intricate, and in many cases indeed baffle all our endeavours to obtain the correct value. These latter cases, which are equally numerous with those whose values we can obtain correctly, arise out of the subject already mentioned in page 123; and will mostly occur towards the end of this chapter: we may, indeed, approximate to their true value by the help of the two lemmata given in the fifth chapter; as will more distinctly appear hereafter.

I would here observe that I have not considered any cases where *more* than three lives are involved: those

cases are so very rare that it would not be worth while to lay down any general rules on the subject; and to investigate them properly would swell the present work to an enormous bulk.

In order to avoid any unnecessary repetitions in the ensuing problems, I will take this opportunity of mentioning, once for all, that I shall in every case denote the given sum by s ; and that the probabilities of living will be still represented by the same quantities as in § 23. The resulting formulæ, which show the value of such sum, will sufficiently enable the experienced analyst to determine its numerical value: but they are too complex and intricate to be inserted as rules, in words at length.

§ 229. I would also observe here, that I use the characters A, B, C to denote the value of an annuity on a life one year *older* than the life A, B , or C respectively: and the characters A', B', C' to denote the value of an annuity on a life one year *younger* than the life A, B , or C respectively. Consequently when the character $A'', B'',$ or C'' occurs, it is meant to denote the value of an annuity on a life n years *older* than A, B , or C respectively: that is, on a life $(n+1)$ years older than A, B , or C respectively.

The same observations will apply to the characters $A'', B'',$ or C'' , which respectively denote the value of an annuity on a life n years *older* than A, B or C : that is, on a life $(n-1)$ years older than A, B , or C ,

respectively. This remark will also extend to the case of such lives considered jointly with any other lives : thus $\overset{\cdot}{A}B$ denotes the value of an annuity on the joint lives of B and a life one year older than A ; $\overset{\cdot}{A}BC$ the value of an annuity on the joint lives of BC and a life one year older than A : and so on.

PROBLEM XXVII.*

§ 230. To determine the present value of a given sum, payable on the decease of A, provided that shall be the *first* which fails of two given lives A, B.

SOLUTION.

The chance of receiving the sum at the end of any one year will depend on the happening of one or other of these two events : 1° that A dies *in* the year, and that B lives to the end of it : 2° that both lives fail *in* the year, restrained however to the contingency that A dies first. The probability that the first event will happen in the first year is $\frac{(a-a')b'}{ab}$; and the probability that the second event will happen in the same period is $\frac{(a-a') \times (b-b')}{2ab}$: these two values, therefore, being added together and multiplied by $s(1+p)^{-1}$, or the present value of the given sum *certain* to be received

* Price, Ques. 11, and Note (M). Dodson, vol. iii. Ques. 23. Simpson's *Sup.* Prob. 32. Morgan, Prob. 16, and in Phil. Trans. for 1788, Prob 2.

at the end of the year, will give $\frac{s}{2}(1+\rho)^{-1} \times$
 $\left(\frac{ab}{ab} - \frac{a'b}{ab} - \frac{a'b}{ab} + \frac{a'b}{ab}\right)$ for the value of the expectation
of receiving such sum at the end of the first year. In
like manner, since the probability that the first event
will happen *in* the second year is $\frac{(a-a'')b''}{ab}$, and the pro-
bability that the second event will happen *in* the same
period is $\frac{(a-a'') \times (b-b'')}{2ab}$, it follows that the sum of these,
multiplied by $s(1+\rho)^{-2}$, will give the value of the ex-
pectation of receiving the sum at the end of the se-
cond year. By a similar method of reasoning we may
find the value of the expectation of receiving the sum
at the end of the third year: and so on for every suc-
ceeding year, to the utmost extent of human life; the
sum of all which yearly values, or the series

$$\begin{aligned} & \frac{s}{2}(1+\rho)^{-1} \times \left[\frac{ab}{ab} - \frac{a'b}{ab} - \frac{a'b}{ab} + \frac{a'b}{ab} \right] + \\ & \frac{s}{2}(1+\rho)^{-2} \times \left[\frac{a'b}{ab} - \frac{a''b''}{ab} - \frac{a''b''}{ab} + \frac{a''b''}{ab} \right] + \\ & \frac{s}{2}(1+\rho)^{-3} \times \left[\frac{a''b''}{ab} - \frac{a'''b'''}{ab} - \frac{a'''b'''}{ab} + \frac{a'''b'''}{ab} \right] + \\ & \qquad \qquad \qquad \&c \qquad \qquad \qquad \&c \qquad \qquad \qquad \&c, \end{aligned}$$

will be the total present value of the sum s to be received on the above contingency.

§ 231. But, the sum of the first two of these perpendicular series (independent of the common multiplier $\frac{s}{2}$) is by Prob. XXII, equal to $\frac{1-\rho AB}{(1+\rho)}$; the

third will be found equal to $-\frac{1+\dot{A}B}{(1+\rho)} \times \frac{\dot{a}}{a}$; and the last, equal to $\dot{A}B \times \frac{\dot{a}}{a}$.* Consequently the total present value of the given sum will be equal to

$$\frac{s}{2} \left[\frac{1-\rho \dot{A}B}{(1+\rho)} - \left(\frac{(1+\dot{A}B)\dot{a}}{(1+\rho)} - \dot{A}B.\dot{a} \right) \frac{1}{a} \right].$$

* The third of these perpendicular series, or $\frac{\dot{a}\dot{b}}{ab(1+\rho)} + \frac{\ddot{a}\ddot{b}}{ab(1+\rho)^2} + \frac{\ddot{\ddot{a}}\ddot{\ddot{b}}}{ab(1+\rho)^3} + \&c$ (by taking all the terms as affirmative and omitting the common multiplier $\frac{s}{2}$), is evidently equal to

$$\frac{\dot{a}\dot{b}}{ab(1+\rho)} \left[1 + \frac{\ddot{a}\dot{b}}{\dot{a}b(1+\rho)} + \frac{\ddot{\ddot{a}}\ddot{b}}{\dot{a}\dot{b}(1+\rho)^2} + \frac{\ddot{\ddot{\ddot{a}}}\ddot{\ddot{\ddot{b}}}}{\dot{a}\dot{b}(1+\rho)^3} + \&c \right]: \text{ which}$$

(since $\frac{\ddot{a}\dot{b}}{\dot{a}b(1+\rho)} + \frac{\ddot{\ddot{a}}\ddot{b}}{\dot{a}\dot{b}(1+\rho)^2} + \frac{\ddot{\ddot{\ddot{a}}}\ddot{\ddot{\ddot{b}}}}{\dot{a}\dot{b}(1+\rho)^3} + \&c$ is equal to $\dot{A}B$) will become equal to $\frac{\dot{a}\dot{b}}{ab(1+\rho)} (1+\dot{A}B) = \frac{1+\dot{A}B}{(1+\rho)} \times \frac{\dot{a}}{a}$. In like manner, it will be evident that the fourth perpendicular series, or

$$\frac{\dot{a}\ddot{b}}{ab(1+\rho)} + \frac{\dot{a}\ddot{\ddot{b}}}{ab(1+\rho)^2} + \frac{\dot{a}\ddot{\ddot{\ddot{b}}}}{ab(1+\rho)^3} + \&c, \text{ is equal to } \frac{\dot{a}}{a} \left[\frac{\dot{a}\ddot{b}}{\dot{a}b(1+\rho)} + \frac{\dot{a}\ddot{\ddot{b}}}{\dot{a}\dot{b}(1+\rho)^2} + \frac{\dot{a}\ddot{\ddot{\ddot{b}}}}{\dot{a}\dot{b}(1+\rho)^3} + \&c \right]: \text{ which (since } \frac{\dot{a}\ddot{b}}{\dot{a}b(1+\rho)} + \frac{\dot{a}\ddot{\ddot{b}}}{\dot{a}\dot{b}(1+\rho)^2} + \frac{\dot{a}\ddot{\ddot{\ddot{b}}}}{\dot{a}\dot{b}(1+\rho)^3} + \&c \text{ is equal to } \dot{A}B) \text{ will become equal}$$

$$\dot{A}B \times \frac{\dot{a}}{a}.$$

The value of these series, however, may be expressed in a different manner, by inverting the method here pursued: for, the perpendicular series is (on the assumption just mentioned)

As this formula will often be referred to in the subsequent problems, it will be convenient to denote it by a more simple expression; therefore let it be represented by A^B : that is, let A^B denote the present value of £1 to be received on the contingency mentioned in the problem; consequently $s \times A^B$ will

$$\begin{aligned} \text{also equal to } \frac{b}{b} \left[\frac{a^1 b}{ab(1+\rho)} + \frac{a^{''} b}{ab(1+\rho)^2} + \frac{a^{'''} b}{ab(1+\rho)^3} + \&c \right] = \\ A^B \times \frac{b}{b}: \text{ and the fourth perpendicular series is also equal to } \\ \frac{a^1 b}{ab(1+\rho)} \left[1 + \frac{a^1 b}{ab(1+\rho)} + \frac{a^{''} b}{ab(1+\rho)^2} + \frac{a^{'''} b}{ab(1+\rho)^3} + \&c \right] = \\ \frac{a^1 b}{ab(1+\rho)} (1 + A^B) = \frac{1 + A^B}{(1+\rho)} \times \frac{b}{b}. \text{ Whence it appears that } \\ \frac{1 + A^B}{(1+\rho)} \times \frac{a^1}{a} = A^B \times \frac{b}{b} \\ \frac{1 + A^B}{(1+\rho)} \times \frac{b}{b} = A^B \times \frac{a^1}{a}: \end{aligned}$$

and consequently, (since *each* series may be summed up in two different ways) that we may adopt either expression for the value of the same: a circumstance of which I have availed myself in some of the subsequent problems. The same observations will apply to any other two lives.

Mr. Morgan (apparently not aware of this property) has run into some strange errors on this subject. He has given *both* expressions in the *same* formula: and such expressions, being *there* always used with *contrary* signs, of course destroy each other; and are therefore *unnecessarily* introduced! These cases are by no means singular, as they occur in almost *all the problems* (involving similar contingencies) inserted by him in the *Philosophical Transactions*. See a remarkable instance of this kind in the *note* to the First Case of Problem XLV.

denote the present value of the given sum under the same circumstances.*

* The only methods of solving this problem (previous to the investigation of it by Mr. Morgan from the real probabilities of life) were those given by Mr. Simpson in the *Supplement* to his *Doctrine of Annuities*, Prob. XXXII, and by Mr. Dodson in the third volume of his *Mathematical Repository*, Ques. XXIII: both of which are deduced from M. De Moivre's hypothesis. Mr. Simpson however has inserted a rule which may be applied to any table of observations: whilst Mr. Dodson still abides by the hypothesis of his friend De Moivre. Mr. Morgan, with his usual antipathy against that theory, has despatched Mr. Dodson's rule in very few words, by saying that "having derived his rule from a wrong hypothesis, he has rendered it of *no use*." (*Phil. Trans.* vol. lxxviii. p. 332): and Dr. Price has observed that he knows of *no other* method of solution but that of Mr. Simpson. Had these gentlemen, however, taken the trouble to compare Mr. Simpson's rule with that given by Mr. Dodson, they would have found that the latter (even on De Moivre's hypothesis) gives the values oftentimes more correctly than the former: and were it adapted to the real probabilities of life, in the same manner as they have adapted Mr. Simpson's, it would in many cases give the values still more corresponding to the *true* values. I agree with them however that neither of these methods can be safely used, except in the middle stages of life: and that it will be best in *all* cases to deduce the values from real observations. Nevertheless as a *near* value may be oftentimes required, without much labour of computation, I have thought it necessary to point out how far these methods may be depended on.

But, is it not singular that, after the unlimited censure which Mr. Morgan has cast upon the "wretched" hypothesis of De Moivre, he should (as editor of Dr. Price's *Observations on Reversionary Payments*) suffer the Fortieth Table in that work to remain without a comment? particularly as he must well know, because he has taken much pains to prove, that the values in that

COROLLARY I.

§ 232. Having thus found the present value of the given sum on the contingency of B surviving A; the present value of the same sum on the contingency of A surviving B (that is, of the same sum to be received on the death of B, provided his life be the first that becomes extinct) is readily found by substituting the symbols $a, \overset{1}{a}, \overset{2}{a}$, &c for $b, \overset{1}{b}, \overset{2}{b}$, &c in the above analysis: whence such value will come out equal to

$$\frac{s}{2} \left[\frac{1 - \rho AB}{(1 + \rho)} - \left(\frac{(1 + AB)\overset{1}{b}}{(1 + \rho)} - AB.\overset{1}{b} \right) \frac{1}{\overset{1}{b}} \right].$$

table are *extremely erroneous*, and in many cases are more than ~~one~~ *third* of their true value too much. And is it not more singular that, to this very hour, it should serve to determine the value of such assurances at the Equitable Society, the business of which Mr. Morgan has so long and so ably conducted! It is true, that the concerns of that society being, for the most part, established upon such fair and truly equitable principles, it little signifies how much is paid for an assurance provided every one pays in proportion: but it must be evident that, in the present case, the rest of the society are benefited at the expense of those who assure on the contingency mentioned in this problem. And it is amusing to observe how blindly the other established Offices, as well as the new ephemeral Companies, have followed this error of their great prototype.

I have inserted the table, above alluded to, at the end of the present work (being the fifty-third table there given); not only as a matter of curiosity, but to show upon what erroneous principles the business of those Offices is sometimes conducted. See the Scholium to Question XXX in Chapter XII.

But, if the present value of the given sum to be received on the death of A according to the problem be once determined, we may easily find the value of the same sum to be received on the death of B, provided his life be the first that fails, by subtracting the value found by the problem from the present value of the given sum to be received on the extinction of the joint lives AB, as found by Prob. XXII: or, in other words, by changing the sign of the second term in the general expression deduced from the problem. Thus $\frac{s}{2} \left[\frac{1-\rho AB}{(1+\rho)} + \left(\frac{(1+AB)a}{(1+\rho)} - AB.a \right) \frac{1}{a} \right]$ will also denote the present value of the given sum on the contingency of A surviving B.*

As these formulæ will be often referred to in the subsequent problems, it will be more convenient to denote them by a more simple expression; therefore let them be represented by B^A : that is, let B^A denote the present value of £1 to be received on the contingency mentioned in this corollary; consequently $s \times B^A$ will denote the present value of the given sum under the same circumstances.

* Universally we shall find that, by this method of investigation, the expression $\frac{s}{2} \left[\frac{1-\rho AB}{(1+\rho)} \mp \left(\frac{(1+AB)a}{(1+\rho)} - AB.a \right) \frac{1}{a} \right] = \frac{s}{2} \left[\frac{1-\rho AB}{(1+\rho)} \pm \left(\frac{(1+AB)b}{(1+\rho)} - AB.b \right) \frac{1}{b} \right]$ will denote the value in either case; where the upper sign is to be taken if the sum depends on the contingency of A's dying first; and the lower sign, if B's dying first.

COROLLARY II.

§ 233. From the principles laid down in the problem we may readily find the value of any sum depending on the contingency of C surviving A; that is, of such sum to be received on the death of A provided his life be the first that becomes extinct of the two lives A, C. The value of such sum will be equal to $\frac{s}{2} \left[\frac{1-\rho AC}{(1+\rho)} - \left(\frac{(1+AC)a'}{(1+\rho)} - AC.a \right) \frac{1}{a} \right]$: and which I shall (for the reasons above mentioned) denote by $s \times A^C$.

In like manner the value of any sum depending on the contingency of C surviving B will be found equal to $\frac{s}{2} \left[\frac{1-\rho BC}{(1+\rho)} - \left(\frac{(1+BC)b'}{(1+\rho)} - BC.b \right) \frac{1}{b} \right]$: which I shall for the same reasons denote by $s \times B^C$. These formulæ will often be referred to in the subsequent problems.

COROLLARY III.

§ 234. If the contingency mentioned in the problem, and on which the given sum is to be received, is *Deferred* for any given number of years ($=n$), less than that to which it is probable the two joint lives may extend, it is evident that the several perpendicular series, in the general expression given in page 184,

must be continued from after the n^{th} term only; and thence to the utmost extent of human life. The sum of which terms will, by an attention to the steps pursued in Problem XXII, cor. 1, be found equal to

$$\frac{s}{2} \left[\frac{1 - \frac{A'B'}{(1+\rho)}}{(1+\rho)} \times \frac{a\beta}{ab} (1+\rho)^{-n} - \frac{1 + \frac{A'B'}{(1+\rho)}}{(1+\rho)} \times \frac{a\beta}{ab} (1+\rho)^{-n} + \frac{A'B'}{(1+\rho)} \times \frac{a\beta}{ab} (1+\rho)^{-n} \right] = \frac{s\beta}{2ab} (1+\rho)^{-(n+1)} \times \left[(1 - \frac{A'B'}{(1+\rho)})_{\alpha} - (1 + \frac{A'B'}{(1+\rho)})_{\alpha} + \frac{A'B'}{(1+\rho)}_{\alpha} \right] : *$$

an expression which I shall denote by $s \times (A^B)^d$.

COROLLARY IV.

§ 235. Or, if the contingency, on which the given sum is to be received, is *Temporary* (that is, to continue only for a given *term* = n , less than that to which it is probable the two joint lives may extend) it is evident that the several perpendicular series, in the general expression given in page 184, must be continued for n terms only: and the sum of these will, by an attention to the steps pursued in Prob. XXII, cor. 2, be found equal to

$$\frac{s}{2} \left[\frac{1 + (AB)^{t-1}}{(1+\rho)} - (AB)^t - \left(\frac{1 + (AB)^{t-1}}{(1+\rho)} \cdot a - (AB)^t \cdot a \right) \frac{1}{a} \right].$$

* It must be remembered that $A'B'$ denotes the value of an annuity on two joint lives, each n years older than A and B ; and that $\frac{A'B'}{(1+\rho)}$ denotes the value of an annuity on two joint lives, each n years older than A and B respectively: agreeably to what has been said in § 229.

But, since $(AB)^{t-1} + \frac{{}_a^i\beta}{ab(1+\rho)^n} = (AB)^t = AB - {}_i^A B \times \frac{{}_a^i\beta}{ab}(1+\rho)^{-n}$, we may (by substituting these values, as explained in the problem just referred to) render this formula more convenient for practice by the following expression $\frac{s}{2} \left[\frac{1-\rho AB}{(1+\rho)} - \frac{1-\rho {}_i^A B}{(1+\rho)} \times \frac{{}_a\beta}{ab}(1+\rho)^{-n} - \frac{1+{}_i^A B}{(1+\rho)} \cdot \frac{{}_a^i}{a} + \frac{1+{}_i^A B}{(1+\rho)} \cdot \frac{{}_a^i\beta}{ab}(1+\rho)^{-n} + {}_i^A B \cdot \frac{{}_a}{a} - {}_i^A B \cdot \frac{{}_a\beta}{ab}(1+\rho)^{-n} \right] = s.A^B - s.(A^B)^d$: that is, equal to the difference between the present value of the sum found by the problem, and the present value of the same sum *Deferred* as in the preceding corollary.*

* I believe that this formula is the first *correct* rule that has ever been given for determining the value of such assurances for *given terms*. In the solution of the fifteenth question in Dr. Price's *Obs: on Rev. Pay.* the author has given a rule by which we may obtain a *near* value of such assurances, and has referred to note H. for a *demonstration* of the same. This demonstration (as he calls it) would indeed lead us to the *exact* value according to any table of observations, agreeably to what the author has justly remarked: the *rule*, however, has not been deduced from this investigation, but from Question XXIII in the third volume of Mr. Dodson's *Mathematical Repository*, or from the process laid down by Mr. Simpson in the *Supplement* to his *Doctrine of Annuities*, page 62.

Mr. Morgan, by way of improving on this *explanation*, has given a rule (without a demonstration) which he says is obtained "without the aid of M. De Moivre's hypothesis." This, however, is not strictly true; for, his rule is deduced on the *supposition* that

COROLLARY V.

§ 256. If the two lives are equal to each other, or of the same age A , the general expression in the problem becomes equal to $\frac{s}{2} \times \frac{1-\rho AA}{(1+\rho)}$; because the second term in such expression, or $\frac{1}{a} \left(\frac{(1+AA)a}{(1+\rho)} - AA.a \right)$, will then vanish altogether: whence it follows that the value will in such case be equal to one

the decrements of life are equal during the given term; similar to the method explained in the note in page 150 of the present work: and this use of M. De Moivre's hypothesis but little accords with what he has repeatedly advanced, "that it is best to exclude it *entirely* from the doctrine of survivorship." This very fact will show that it may occasionally be used with good effect, even in the solution of the very problem where he has censured it the most. I shall here subjoin an investigation of Mr. Morgan's rule, which will show the method of proceeding in similar cases, when we wish to adopt the same principles.

It will appear that the general series in the problem may be converted into the following one,

$$\begin{aligned} & \frac{s}{2(1+\rho)} \times \frac{a-a'}{a} \left(\frac{b}{b} + \frac{b'}{b} \right) + \\ & \frac{s}{2(1+\rho)^2} \times \frac{a-a''}{a} \left(\frac{b}{b} + \frac{b''}{b} \right) + \\ & \frac{s}{2(1+\rho)^3} \times \frac{a-a'''}{a} \left(\frac{b}{b} + \frac{b'''}{b} \right) + \\ & \qquad \qquad \qquad \&c \qquad \qquad \qquad \&c. \end{aligned}$$

Now, if we suppose the decrements of A 's life to be equal during the given term, (which in many cases will be found to correspond very neatly with the fact,) and if such decrements be de-

half the present value of the given sum to be received on the extinction of the joint lives.

This observation will equally apply to the formulæ, in the last two corollaries, for *terms*; and is indeed self-evident. For, as the given sum is not to be re-

noted by $\delta = \frac{a-\alpha}{n}$; then will the first n terms of this series be equal to

$$\begin{aligned} & \frac{s.\delta}{2a(1+\rho)} \left(\frac{b}{b} + \frac{b}{b} \right) + \\ & \frac{s.\delta}{2a(1+\rho)^2} \left(\frac{b}{b} + \frac{b}{b} \right) + \\ & \frac{s.\delta}{2a(1+\rho)^3} \left(\frac{b}{b} + \frac{b}{b} \right) + \\ & \dots \dots \dots \\ & \frac{s.\delta}{2a(1+\rho)^n} \left(\frac{\beta}{b} + \frac{\beta}{b} \right). \end{aligned}$$

But, the first of these perpendicular series, within the parentheses, is (independent of the common multiple $\frac{s.\delta}{2a}$) equal to $\frac{1+(B)^{t-1}}{(1+\rho)}$; and the second is, in like manner, equal to $(B)^t$: whence the total present value of the assurance for the given term, will upon this hypothesis be denoted by $\frac{s.\delta}{2a} \left[\frac{1+(B)^{t-1}}{(1+\rho)} + (B)^t \right] = \frac{s(a-\alpha)}{2an} \left[\frac{1+(B)^{t-1}}{(1+\rho)} + (B)^t \right]$: and whence Mr. Morgan's rule is manifest.

But this rule may be more conveniently expressed by $\frac{s(a-\alpha)}{2an} \times \frac{1+(2+\rho)B - [1+(2+\rho)B^n] \times \frac{\beta}{b}(1+\rho)^{-n}}{(1+\rho)}$: which, when the two series are continued to the utmost extent of human life (~~B~~ being the oldest), will become $\frac{s(a-\alpha)}{2an} \times \frac{1+(2+\rho)B}{1+\rho}$. In this ~~last~~

ceived till after the extinction of the joint lives, and as it is certainly an *even* chance which of the two may survive, it is manifest that one *half* the present value of the sum to be received on the extinction of those joint lives will be the true value of the sum depending on this contingency; whether such contingency extends to the whole continuance of the given lives, or exists only for a given term.

Observations on Mr. Morgan's Method of investigating this Problem.

§ 237. I cannot dismiss this problem without some remarks on the singular and unscientific manner in which Mr. Morgan has summed up the four collateral series (in page 184) which express the present value of the given sum to be received on the contingency mentioned in the problem : and I am the more induced to do this because similar expressions arise in many of his subsequent problems, in which the same loose and eccentric method has been pursued ; whereby an air of obscurity and difficulty has been thrown over the subject, which it is my intention here

ter case, however, the formula cannot be depended upon, unless A is much the youngest of the two lives ; but at the same time not younger than 15 years of age : indeed it must at *all* times be used with proper caution, and with a due regard to the tables of observations made use of. These remarks will equally apply to the solution of the sixteenth question in Dr. Price's work above quoted.

to remove. Mr. Morgan divides the whole expression into the two following collateral series,* viz.

$$\begin{aligned} & \frac{s}{2(1+p)} \left[\frac{(a-a')b}{ab} + \frac{(a-a')b}{ab} \right] + \\ & \frac{s}{2(1+p)^2} \left[\frac{(a'-a'')b''}{ab} + \frac{(a'-a'')b''}{ab} \right] + \\ & \frac{s}{2(1+p)^3} \left[\frac{(a''-a''')b'''}{ab} + \frac{(a''-a''')b'''}{ab} \right] + \\ & \qquad \qquad \qquad \&c \qquad \qquad \qquad \&c; \end{aligned}$$

which is evidently the same series as that which I have given in page 184.

* In the subsequent process given in the text I have adopted the method of notation made use of throughout the whole of this work, in order to prevent any obscurity or circumlocution by the introduction of new symbols. But, lest I should be charged with having misrepresented Mr. Morgan, I shall here subjoin the method which he himself has pursued in investigating this problem; and for the sake of greater accuracy shall adopt the same characters which he has used.

Let r denote £1 increased by its interest for a year: let a represent the number of persons living in the table at the age of A; and let $a', a'', a''', \&c$, represent the decrements at the end of the first, second, third, &c, years from the age of A. Let b represent the number of persons living at the age of B; and let $c, d, e, f, \&c$, represent the number of persons living at the end of the first, second, third, fourth, &c, years from the age of B.

Then (in the words of Mr. Morgan, in *Phil. Trans.* for 1788, p. 342) it will follow that the present value of S to be received on the death of A, should that happen in the life-time of B, will be

expressed by the series $S \times \left[\frac{(b+c)a'}{2abr} + \frac{(c+d)a''}{2abr^2} + \frac{(d+e)a'''}{2abr^3} + \frac{(e+f)a'''}{2abr^4} + \&c \right]$. This series may be resolved into the two fol-

§ 238. The first of these collateral series he expands into a more complex one, by making it equal to

$$\begin{aligned} & \frac{s}{2(1+\rho)} \left[\frac{b}{b} - \frac{a'b}{ab} \dots \dots \right] + \\ & \frac{s}{2(1+\rho)^2} \left[\frac{b''}{b} - \frac{a''b''}{ab} - \frac{b'}{b} + \frac{a'b'}{ab} \right] + \\ & \frac{s}{2(1+\rho)^3} \left[\frac{b'''}{b} - \frac{a'''b'''}{ab} - \frac{b''}{b} + \frac{a''b''}{ab} \right] + \\ & \qquad \qquad \qquad \&c \qquad \qquad \&c \qquad \qquad \&c; \end{aligned}$$

and he then proceeds to sum up these several collateral series in the following manner. The first he makes (independent of the common multiple $\frac{s}{2}$) equal to B , the second equal to $-AB$, the third equal to $-\frac{B}{(1+\rho)} \times \frac{b}{b}$, and the fourth equal to $\frac{AB}{(1+\rho)} \times \frac{b}{b}$:

lowing; $\frac{s}{2} \times \left[\frac{ca'}{abr} + \frac{da''}{abr^2} + \frac{ea'''}{abr^3} + \frac{fa'''}{abr^4} + \&c \right] + \frac{s}{2} \times$
 $\left[\frac{ba'}{abr} + \frac{ca''}{abr^2} + \frac{da'''}{abr^3} + \frac{ea'''}{abr^4} + \&c \right]$. The first of these two
 may be again resolved into $\frac{s}{2} \times \left[\frac{c}{br} - \frac{ca-ca'}{abr} + \frac{d}{br^2} - \right.$
 $\left. \frac{da-da'}{abr^2} + \frac{e}{br^3} - \frac{ea-ea'-e''a''-e'''a'''}{abr^3} + \&c \right] - \frac{sc}{2br} \times$
 $\left[\frac{d}{cr} - \frac{da-da'}{acr} + \frac{e}{cr^2} - \frac{ea-ea'-e''a''-e'''a'''}{acr^2} + \&c \right]$. Let B denote
 the value of an annuity on the life of B , C the value of an annuity
 on a life one year older than B , AB and AC the values of annuities
 on the joint lives of A and B and of A and C , and these series
 will be $= \frac{S \times (B-AB)}{2} - \frac{S \times c(C-AC)}{2br}$. Again, the second
 series above mentioned, or $\frac{s}{2} \times \left[\frac{ba'}{abr} + \frac{ca''}{abr^2} + \frac{da'''}{abr^3} + \&c \right]$, by

so that the total value of these four collateral series (or of the *first* collateral series in page 196) is made

$$\text{equal to } B - \frac{B}{(1+\rho)} \times \frac{b}{b} + \frac{AB}{(1+\rho)} \times \frac{b}{b} - AB.$$

And, by proceeding in a similar manner with respect to the *second* collateral series in page 196, the sum

$$\text{of it is found equal to } B \times \frac{b}{b} - \frac{B}{(1+\rho)} - AB \times \frac{b}{b} + \frac{AB}{(1+\rho)}.$$

Consequently, the total present value of the given sum, to be received on the contingency

$$\text{mentioned in the problem, is equal to } \frac{s}{2} \left[B \times \frac{b}{b} - \frac{B}{(1+\rho)} + \frac{AB}{(1+\rho)} - AB + B - \frac{B}{(1+\rho)} \times \frac{b}{b} + \frac{AB}{(1+\rho)} - \frac{b}{b} - AB \times \frac{b}{b} \right].$$

pursuing the same steps may be found = $\frac{\beta S}{2b} (K - AK) - \frac{S \times (B - AB)}{2r}$, where β denotes the number of persons living

at the age of a person one year younger than B, K the value of an annuity on that life, and AK the value of an annuity on the joint lives of A and K. The whole value of the survivorship is therefore = $S \times \left[\frac{(r-1) \times (B-AB)}{2r} + \frac{\beta(K-AK)}{2b} - \frac{c(C-AC)}{2br} \right]$.

These are Mr. Morgan's own words: and if the several values, above alluded to, be expressed by the characters which I have adopted in the present work, they will be found to agree with the process given in the text. But $B - \frac{c.C}{br} = \frac{c}{br}$, and $\frac{\beta.K}{b} -$

$\frac{B}{r} = \frac{1}{r}$; whence the absurdity of his formula. Mr. Morgan however has at length altered his formula for this particular case; although he still preserves the same method of solution, both here and in all his subsequent problems. See Price's *Obs.* edit. 1803, note (O).

§ 239. But, surely it cannot be necessary to tell Mr. Morgan that $B - \frac{B}{(1+\rho)} \times \frac{b}{b}$ is equal to $\frac{1}{(1+\rho)} \times \frac{b}{b}$;* and that $B \times \frac{b}{b} - \frac{B}{(1+\rho)}$ is equal to $\frac{1}{(1+\rho)}$ only!!! Whereby the complex and troublesome formula, above given, may be more simply expressed by $\frac{s}{2} \left[\frac{1-\rho AB}{(1+\rho)} + \frac{1+AB}{(1+\rho)} \times \frac{b}{b} - AB \times \frac{b}{b} \right]$, and which is one of the formulæ which I have given, in the note in page 189, for the value of the sum depending on this contingency. Indeed it is difficult to conceive why he should ever have thought it necessary to employ those quantities which denote the value of an annuity on a *single* life; since it is clear from the nature of the several series in page 196, that such values could not properly belong to the subject; and that no quantities are involved in the solution but such as arise from *two joint* lives.

As this strange error runs through the whole of Mr. Morgan's papers in the Philosophical Transactions, where this and similar contingencies are involved, I have thought it right to take this public notice of it;

* This is indeed evident on inspection of the first and third of the collateral series in page 197. For, it is manifest that all the subsequent terms of those two series *after the first term*, destroy each other; and consequently that the sum of those two collateral series will be equal to the first term only: that is, (independent of the common multiple $\frac{s}{2}$) equal to $\frac{1}{(1+\rho)} \times \frac{b}{b}$. In like manner we may prove the equality of the other two quantities alluded to in the text.

not only with a view to obviate any difficulty that may arise from a comparison of his formulæ with those which I have here deduced, but to clear the subject from all diffuse and unconnected matter. And I shall have occasion to make a similar remark on his method of summing the several series where *three* lives are concerned: for which, see the *Observations* in § 251 &c, at the end of Problem XXIX; and in § 272 at the end of Problem XXXV.

PROBLEM XXVIII.*

§ 240. To determine the present value of a given sum payable on the decease of A, provided that shall be the *second* which fails of two given lives A, B.

SOLUTION.

It is evident, in this case, that the chance of receiving the sum at the end of the first year will depend only on one contingency; viz. that both the lives become extinct in that period, restrained however to the condition that A dies last, the probability of which in the first year is $\frac{(a-a') \times (b-b')}{2ab}$; and which,

* Price, Ques. 12, and Note (M). Dodson, vol. iii. Ques. 24. Simpson, Prob. 28; and *Sup.* Prob. 33; also Prob. 3 in page 72. De Moivre, Prob. 17. Morgan, Prob. 17; and in *Phil. Trans.* for 1788, Prob. 3, page 347.

being multiplied by $s(1+\rho)^{-1}$, or the present value of the given sum, certain to be received at the end of the year, will give $\frac{s}{2}(1+\rho)^{-1} \times (\frac{a^1 b}{a b} + \frac{a^1 b}{a b} - \frac{a^1 b}{a b} - \frac{a^1 b}{a b})$ for the expectation of receiving such sum at the end of the first year.

In the second and following years, however, the chance of receiving the sum will depend on the happening of either of two events: 1° that both the lives fail in the year, A having died last: 2° that only A dies in the year, and that B dies in either of the preceding years. The probability that the first event will happen in the second year is $\frac{(a''-a') \times (b''-b')}{2ab}$; and the probability of the second event happening in the same period is $\frac{a''-a'}{a}(1-\frac{b'}{b})$. These two values therefore, being added together and multiplied by $s(1+\rho)^{-2}$, will give $\frac{s}{2}(1+\rho)^{-2} \times (\frac{2a''}{a} - \frac{2a'}{a} - \frac{a^1 b}{a b} + \frac{a^1 b}{a b} + \frac{a^1 b}{a b} - \frac{a^1 b}{a b})$ for the expectation of receiving the given sum at the end of the second year.

In like manner, it will be found that the probability of the first event happening in the third year is $\frac{(a'''-a'') \times (b'''-b'')}{2ab}$; and the probability that the second event will happen in the same period is $\frac{a'''-a''}{a}(1-\frac{b''}{b})$; therefore these two values being added together and multiplied by $s(1+\rho)^{-3}$, will give $\frac{s}{2}(1+\rho)^{-3} \times (\frac{2a'''}{a} - \frac{2a''}{a} - \frac{a^1 b}{a b} + \frac{a^1 b}{a b} + \frac{a^1 b}{a b} - \frac{a^1 b}{a b})$ for the ex-

pectation of receiving the sum at the end of the third year: and so on for every subsequent year, to the utmost extent of human life; the sum of all which yearly values, or the series*

$$\begin{aligned} & \frac{s}{2}(1+\rho)^{-1} \times \left[\frac{2a}{a} - \frac{2a'}{a} - \frac{ab}{ab} + \frac{a'b}{ab} + \frac{a'b}{ab} - \frac{a'b}{ab} \right] + \\ & \frac{s}{2}(1+\rho)^{-2} \times \left[\frac{2a'}{a} - \frac{2a''}{a} - \frac{a'b'}{ab} + \frac{a''b'}{ab} + \frac{a''b'}{ab} - \frac{a''b'}{ab} \right] + \\ & \frac{s}{2}(1+\rho)^{-3} \times \left[\frac{2a''}{a} - \frac{2a'''}{a} - \frac{a'b''}{ab} + \frac{a'''b''}{ab} + \frac{a'''b''}{ab} - \frac{a'''b''}{ab} \right] + \\ & \qquad \qquad \qquad \&c \qquad \qquad \qquad \&c \qquad \qquad \qquad \&c, \end{aligned}$$

will be the total present value of the sum s to be received on the above contingency.

§ 241. But the sum of the first two of these perpendicular series is, by Prob. XXII, equal to the present value of the given sum to be received on the decrease of A ; that is, equal to $s \times \frac{1-\rho A}{(1+\rho)}$: and the remaining four of these perpendicular series are the same as those produced in the last problem, with a contrary sign; and the sum of which is there denoted by

* The value of the first year's expectation has been found in the preceding part of the investigation, equal to $\frac{s}{2}(1+\rho)^{-1} \times \left(\frac{ab}{ab} + \frac{a'b}{ab} - \frac{a'b}{ab} - \frac{a'b}{ab} \right)$; an expression which is evidently equal to $\frac{s}{2}(1+\rho)^{-1} \times \left(\frac{2a}{a} - \frac{2a'}{a} - \frac{ab}{ab} + \frac{a'b}{ab} + \frac{a'b}{ab} - \frac{a'b}{ab} \right)$. This latter value has been assumed in order to correspond with the terms of the subsequent expectations, and thereby render the several collateral series complete. A similar method is pursued in many of the following problems.

$\frac{s}{2} \left[\frac{1-\rho AB}{(1+\rho)} - \left(\frac{(1+AB)a}{(1+\rho)} - AB.a \right) \frac{1}{a} \right]$. Consequently the total value of these several series will be expressed by $s \left[\frac{1-\rho A}{(1+\rho)} - A^B \right]$.

COROLLARY I.

§ 242. Having thus found the present value of the given sum on the contingency of A dying after B; the present value of the same sum on the contingency of B dying after A (that is, of the same sum to be received on the death of B, provided his life is the second that fails) is readily determined by subtracting the value above found from the present value of the given sum to be received on the extinction of the longest of the two lives; which, by Prob. XXII, cor. 3, is expressed by $s \times \frac{1-\rho(A+B-AB)}{(1+\rho)}$. Consequently, the present value of the given sum, on the contingency of B dying after A, will be equal to $\frac{s}{2} \left[\frac{1-\rho(2B-AB)}{(1+\rho)} - \left(\frac{(1+AB)a}{(1+\rho)} - AB.a \right) \frac{1}{a} \right] = s \left[\frac{1-\rho B}{(1+\rho)} - B^A \right]$.

COROLLARY II.

§ 243. If the contingency on which the sum is to be received is *Deferred* for a given number of years ($=n$), less than that to which it is probable the life A may extend, it is evident that the sum of all the terms of the first two perpendicular series, after the n^{th} year, will, by Prob. XXII, cor. 1, be equal to $s \times \frac{1-\rho A^n}{(1+\rho)} \times \frac{a}{a} (1+\rho)^{-n}$; and the sum of all the

similar terms of the remaining four perpendicular series will be denoted by the same expression as that which has been deduced in the second corollary to the preceding problem, but with a contrary sign. Therefore if the value found by that corollary be subtracted from the present value of the assurance of the given sum after the extinction of the life A, provided that happens after the given term, the difference will be the value required.

COROLLARY III.

§ 244. If the contingency on which this sum is to be received commences immediately, but continues only for a certain number of years ($= n$), less than that to which it is probable the life A may extend: or, in other words, if we wish to determine the value of a *Temporary* assurance of such sum; it is evident that the several perpendicular series given in the problem, must be continued only for n terms. Now, the sum of the first n terms of the first two of these perpendicular series will, by Prob. XXII, cor. 2, be found equal to $\frac{1-\rho A}{(1+\rho)} - \frac{1-\rho A^0}{(1+\rho)} \times \frac{a}{a}(1+\rho)^{-n}$: from which, if we subtract the first n terms* of the four

* When n is equal to, or greater than, the number of years between the age of B and that age in the table of observations at which human life becomes extinct, we must subtract the *whole* value of the four remaining series as found by the last problem. That is, the correct value in both such cases will be denoted by &

multiplied into
$$\frac{1-\rho A - (1-\rho A^0) \frac{a}{a} (1+\rho)^{-n}}{(1+\rho)} - A^B.$$

remaining series, as found by cor. 4 in the preceding problem, the difference will be the value of the assurance for the given term.*

* See the note to Cor. 4 in the last problem; where the process there pursued will enable us to find a *near* value of an assurance for a given *term*, according to the conditions of the present problem. For, if we deduct the near value, there deducted, from the near value of an assurance of the same sum on the life of A for the same term, as found by the process laid down in the note to Prob. XXII, cor. 2, the difference, or $\frac{s \cdot \delta}{2a} \left[2 \times \frac{1 - (1 + \rho)^{-n}}{\rho} - \frac{1 + (B)^{t-1}}{(1 + \rho)} - (B)^t \right]$, will be the value required.

This formula is the same, in fact, as that given by Mr. Morgan in Note H of Dr. Price's *Obs. on Rev. Pay.*; and is another instance of the utility of De Moivre's hypothesis: but, it may be rendered more convenient for practice by the substitutions adopted in the note to Cor. 4 of the preceding problem; whereby it will become $\frac{s(a-\alpha)}{2an} \left[2 \times \frac{1 - (1 + \rho)^{-n}}{\rho} - \frac{1 + (2 + \rho)B - [1 + (2 + \rho)B^2] \times \frac{\beta}{b} (1 + \rho)^{-n}}{(1 + \rho)} \right]$.

When n is equal to, or greater than, the number of years between the age of B and that age in the table of observations when human life becomes extinct, β is equal to 0; and consequently $[1 + (2 + \rho)B^2] \times \frac{\beta}{b} (1 + \rho)^{-n}$ vanishes: whereby the formula in both these cases would become $\frac{s(a-\alpha)}{2an} \left[2 \times \frac{1 - (1 + \rho)^{-n}}{\rho} - \frac{1 + (2 + \rho)B}{(1 + \rho)} \right]$. But in all *such* cases the value is best obtained by the rule given in the preceding note: since it is then equal to the difference between the value of an assurance on the life A for the given *term*, and the value of an assurance on the *whole* life A provided he be the first that fails.

COROLLARY IV.

§ 245. If the two lives are equal, or of the same age A , the general expression in the problem becomes (agreeably to what has been already said in cor. 5 of the last problem) equal to $\frac{s}{2} \times \frac{1 - \rho(2A - AA)}{(1 + \rho)}$: that is, equal to one half the present value of the given sum to be received on the extinction of the longest of the two lives.

PROBLEM XXIX.*

§ 246. To determine the present value of a given sum payable on the decease of A , provided he be the first that fails of three given lives A, B, C .

SOLUTION.

In order to receive the given sum at the end of any one year, it is necessary that one or other of four different events should happen: 1° that all the three lives fail in the year, A having died first: 2° that A and B fail in the year, A having died first, and that C lives to the end of it: 3° that A and C fail in the year, A having died first, and that B lives to the end of it: 4° that only A dies in the year, and that B and C both live to the end of it. The probabilities of the happening of these several events in the first year are re-

* Dodson, vol. iii. Ques. 32. Morgan, Prob. 19; and in Phil. Trans. for 1791, Prob. 1, page 248.

spectively $\frac{(a-a')(b-b')(c-c')}{3abc}$, $\frac{(a-a')(b-b')c'}{2abc}$, $\frac{(a-a')(c-c')b'}{2abc}$,
and $\frac{(a-a')b'c'}{abc}$; which being added together and multi-
plied by $s(1+\rho)^{-1}$, will give $\frac{s}{6}(1+\rho)^{-1} \times \left(\frac{2abc}{abc} - \frac{2a'b'c'}{abc} - \frac{2a'b'c'}{abc} + \frac{2a'b'c'}{abc} + \frac{a'b'c'}{abc} - \frac{a'b'c'}{abc} + \frac{a'b'c'}{abc} - \frac{a'b'c'}{abc} \right)$
for the expectation of receiving the sum at the end of
the first year.

In like manner, it will be found that the probabilities
of the happening of these several events in the second
year are respectively $\frac{(a'-a'')(b'-b'')(c'-c'')}{3abc}$, $\frac{(a'-a'')(b'-b'')c''}{2abc}$,
 $\frac{(a'-a'')(c'-c'')b''}{2abc}$, and $\frac{(a'-a'')b''c''}{abc}$: which being added toge-
ther, and multiplied by $s(1+\rho)^{-2}$, will give the ex-
pectation of receiving the sum at the end of the second
year. And by a similar process we may find the ex-
pectation of receiving the sum at the end of the third
and every succeeding year, to the utmost extent of
human life; the sum of all which yearly values, or
the series

$$\begin{aligned} &= \left[\frac{2abc}{abc} - \frac{2a'b'c'}{abc} - \frac{2a'b'c'}{abc} + \frac{2a'b'c'}{abc} + \frac{a'b'c'}{abc} - \frac{a'b'c'}{abc} + \frac{a'b'c'}{abc} - \frac{a'b'c'}{abc} \right] + \\ &= \left[\frac{2a'b'c'}{abc} - \frac{2a''b''c''}{abc} - \frac{2a''b''c''}{abc} + \frac{2a''b''c''}{abc} + \frac{a''b''c''}{abc} - \frac{a''b''c''}{abc} + \frac{a''b''c''}{abc} - \frac{a''b''c''}{abc} \right] + \\ &= \left[\frac{2a''b''c''}{abc} - \frac{2a'''b'''c'''}{abc} - \frac{2a'''b'''c'''}{abc} + \frac{2a'''b'''c'''}{abc} + \frac{a'''b'''c'''}{abc} - \frac{a'''b'''c'''}{abc} + \frac{a'''b'''c'''}{abc} - \frac{a'''b'''c'''}{abc} \right] + \\ &\quad \&c \qquad \&c \qquad \&c \qquad \&c, \end{aligned}$$

Will be the total present value of the given sum to be
received on the contingency above mentioned.

§ 247. But the sums of these eight perpendicular and collateral series (independent of the common multiple s) are respectively equal to $\frac{1+ABC}{3(1+\rho)} - \frac{ABC}{3} - \frac{1+ABC}{3(1+\rho)} \cdot \frac{a}{a} + \frac{ABC}{3} \cdot \frac{a}{3a} + \frac{1+ABC}{6(1+\rho)} \cdot \frac{b}{b} - \frac{ABC}{6} \cdot \frac{b}{6b} + \frac{1+ABC}{6(1+\rho)} \cdot \frac{c}{c} - \frac{ABC}{6} \cdot \frac{c}{6c}$. * Whence, the sum of the

* The sum of the first and second of these collateral series is found by the process laid down in Prob. XXII; and the remaining ones, in the following manner. The sum of the third perpendicular series, or

$\frac{abc}{abc(1+\rho)} + \frac{abc}{abc(1+\rho)^2} + \frac{abc}{abc(1+\rho)^3} + \&c$ (by taking all the terms as affirmative and leaving out the common multiple $\frac{2s}{6}$) is evidently equal to $\frac{abc}{abc(1+\rho)} \left[1 + \frac{abc}{abc(1+\rho)} + \frac{abc}{abc(1+\rho)^2} + \frac{abc}{abc(1+\rho)^3} + \&c \right]$: which, since $\frac{abc}{abc(1+\rho)} + \frac{abc}{abc(1+\rho)^2} + \frac{abc}{abc(1+\rho)^3} + \&c$ is equal to ABC , will become

equal to $\frac{abc}{abc(1+\rho)} (1+ABC) = \frac{1+ABC}{(1+\rho)} \times \frac{a}{a}$. In like manner, it is evident that, by leaving out the common multiple $\frac{2s}{6}$, the fourth perpendicular series, or $\frac{abc}{abc(1+\rho)} + \frac{abc}{abc(1+\rho)^2} + \frac{abc}{abc(1+\rho)^3} + \&c$, is equal to $\frac{a}{a} \left[\frac{abc}{abc(1+\rho)} + \frac{abc}{abc(1+\rho)^2} + \frac{abc}{abc(1+\rho)^3} + \&c \right]$: which, since $\frac{abc}{abc(1+\rho)} + \frac{abc}{abc(1+\rho)^2} + \frac{abc}{abc(1+\rho)^3} + \&c$ is equal to ABC , will become equal

first and second is (as in Prob. XXII) equal to $\frac{s}{3} \times \frac{1 - \rho ABC}{(1 + \rho)}$; the sum of the third and fourth is equal to $-\frac{s}{3a} \left[\frac{(1 + ABC)a}{(1 + \rho)} - ABC \cdot a \right]$; the sum of the fifth

$ABC \times \frac{a}{a}$. Therefore the sum of these two series, taking the former of them as negative, agreeably to the general expression in the text, will be $-\frac{1 + ABC}{(1 + \rho)} \times \frac{a}{a} + ABC \times \frac{a}{a}$; and which being multiplied by their common multiple $\frac{2s}{6}$, will produce $-\frac{s}{3a} \times \left[\frac{(1 + ABC)a}{(1 + \rho)} - ABC \cdot a \right]$ for the value of the same. The same method must be pursued in order to find the sum of the next two series, and also of the last two: but enough has been here said to enable the reader to perform these operations, without stating the process at large.

I would, however, observe that these several series may be expressed by other formulæ than those given in the text: for, the third perpendicular series, taking all the terms as affirmative and leaving out the common multiple $\frac{2s}{6}$, is also equal to $\frac{bc}{bc} \times$

$$\left[\frac{abc}{abc(1 + \rho)} + \frac{abc}{abc(1 + \rho)^2} + \frac{abc}{abc(1 + \rho)^3} + \&c \right]; \text{ which (since}$$

the sum of the terms within the brackets is equal to ABC) will

be equal to $ABC \times \frac{bc}{bc}$: and the fourth perpendicular series,

leaving out the common multiple $\frac{2s}{6}$, is also equal to $\frac{abc}{abc(1 + \rho)} \times$

$$\left[\frac{abc}{abc(1 + \rho)} + \frac{abc}{abc(1 + \rho)^2} + \frac{abc}{abc(1 + \rho)^3} + \&c \right]: \text{ which}$$

and sixth is equal to $\frac{s}{6b} \left[\frac{(1+ABC)b}{(1+\rho)} - ABC.b \right]$; and the sum of the seventh and eighth is equal to

(since the sum of the terms within the brackets is equal to $1+ABC$) becomes equal to $\frac{1+ABC}{(1+\rho)} \times \frac{b}{bc}$. Consequently, the sum of the third and fourth perpendicular series given in the text will also be denoted by $+\frac{s}{3bc} \left[\frac{(1+ABC)b}{(1+\rho)} - ABC.b \right]$; and may be substituted at pleasure for the same. And universally we shall find that

$$\begin{aligned} \frac{1+ABC}{(1+\rho)} \cdot \frac{a}{a} &= ABC \cdot \frac{b}{bc} \\ ABC \cdot \frac{a}{a} &= \frac{1+ABC}{(1+\rho)} \cdot \frac{b}{bc} \\ \frac{1+ABC}{(1+\rho)} \cdot \frac{b}{b} &= ABC \cdot \frac{a}{ac} \\ ABC \cdot \frac{b}{b} &= \frac{1+ABC}{(1+\rho)} \cdot \frac{a}{ac} \\ \frac{1+ABC}{(1+\rho)} \cdot \frac{c}{c} &= ABC \cdot \frac{a}{ab} \\ ABC \cdot \frac{c}{c} &= \frac{1+ABC}{(1+\rho)} \cdot \frac{a}{ab} \end{aligned}$$

It therefore appears that *each* series may be summed up in *two* different ways, and that we may adopt either mode of expression for the value of the same. For the sake of uniformity I have, in this and the following problems, kept to those which are given in the text: but Mr. Morgan (by merely changing these expressions, one for the other, according to the seniority of the lives, and then treating them as *different* quantities,) has thrown an air of obscurity and confusion throughout the whole of his investigations; as I shall point out in the *Observations* at the end of this problem. See § 251.

$$\frac{s}{6c} \left[\frac{(1 + \dot{ABC})'c}{(1 + \rho)} - \dot{ABC}.c \right]. \text{ Consequently, the total present value of the given sum to be received on the above contingency, will be equal to } s \text{ multiplied into}$$

$$\frac{1 - \rho \dot{ABC}}{3(1 + \rho)} - \frac{1}{3a} \left[\frac{(1 + \dot{ABC})'a}{(1 + \rho)} - \dot{ABC}.a \right] + \frac{1}{6b} \times$$

$$\left[\frac{(1 + \dot{ABC})'b}{(1 + \rho)} - \dot{ABC}.b \right] + \frac{1}{6c} \left[\frac{(1 + \dot{ABC})'c}{(1 + \rho)} - \dot{ABC}.c \right].$$

As this formula will often be referred to, in the subsequent problems, it will be convenient to denote it by a more simple expression; therefore let it be represented by $\mathcal{A}^{BC}$: that is, let $\mathcal{A}^{BC}$ denote the present value of £1 to be received on the above contingency: consequently $s \times \mathcal{A}^{BC}$ will denote the present value of the given sum under the same circumstances.

COROLLARY I.

§ 248. If it were required to find the present value of the given sum payable on the decease of B, provided he be the first that fails of the three lives, we may readily obtain such value by substituting A for B, and B for A, in the investigation of the problem. Whence, the present value required would come out equal to s multiplied into $\frac{1 - \rho \dot{ABC}}{3(1 + \rho)} +$

$$\frac{1}{6a} \left[\frac{(1 + \dot{ABC})'a}{(1 + \rho)} - \dot{ABC}.a \right] - \frac{1}{3b} \left[\frac{(1 + \dot{ABC})'b}{(1 + \rho)} - \dot{ABC}.b \right]$$

$$ABC.b \Big] + \frac{1}{6c} \left[\frac{(1+ABC)c}{(1+\rho)} - ABC.c \right] = B^{AC}.*$$

By a similar process it will be found that the present value of the given sum payable on the decease of C, provided he be the first that fails of the three lives, is

$$\text{equal to } s \text{ multiplied into } \frac{1-\rho ABC}{3(1+\rho)} + \frac{1}{6a} \left[\frac{(1+ABC)a}{(1+\rho)} - ABC.a \right] + \frac{1}{6b} \left[\frac{(1+ABC)b}{(1+\rho)} - ABC.b \right] - \frac{1}{3c} \times \left[\frac{(1+ABC)c}{(1+\rho)} - ABC.c \right] = C^{AB}.*$$

As these formulæ may occasionally be found of use, I have thought proper to insert them here.

COROLLARY II.

§ 249. If the three lives are equal, or of the same

* I have represented these complex formulæ by the more simple quantities B^{AC} and C^{AB} , for the sake of a more convenient reference; and the following process will show with how little trouble they may be converted into each other.—Let us make

$$\frac{1}{a} \left[\frac{(1+ABC)a}{(1+\rho)} - ABC.a \right] = a, \quad \frac{1}{b} \left[\frac{(1+ABC)b}{(1+\rho)} - ABC.b \right]$$

$$= b, \text{ and } \frac{1}{c} \left[\frac{(1+ABC)c}{(1+\rho)} - ABC.c \right] = c; \text{ then will}$$

$$A^{BC} = \frac{1-\rho ABC}{3(1+\rho)} - \frac{a}{3} + \frac{b}{6} + \frac{c}{6}$$

$$B^{AC} = \frac{1-\rho ABC}{3(1+\rho)} + \frac{a}{6} - \frac{b}{3} + \frac{c}{6}$$

$$C^{AB} = \frac{1-\rho ABC}{3(1+\rho)} + \frac{a}{6} + \frac{b}{6} - \frac{c}{3}.$$

age A , the last three terms in each of the above expressions destroy each other; and the formula is then reduced to $\frac{s}{3} \times \frac{1-\rho AAA}{(1+\rho)}$: an expression which denotes one third of the present value of the given sum to be received on the extinction of the three joint lives.

COROLLARY III.*

§ 250. If the contingency, on which the sum is to be received, continues only for a given term ($=n$), the present value of such sum will be equal to the sum of the first n terms of the several series given in the problem: the method of determining which will be manifest from the many examples which have preceded.†

* Morgan, Prob. 31.

† We may in general obtain a near value of the sum, in this case, by the help of M. De Moivre's hypothesis, which will save much time and trouble. For, the series given in the problem is evidently equal to

$$\begin{aligned} & \frac{s}{6(1+\rho)} \times \frac{a-a'}{a} \left[\frac{2bc}{bc} + \frac{2b'c'}{bc} + \frac{b'c}{bc} + \frac{b'c'}{bc} \right] + \\ & \frac{s}{6(1+\rho)^2} \times \frac{a'-a''}{a} \left[\frac{2b'c'}{bc} + \frac{2b''c''}{bc} + \frac{b''c'}{bc} + \frac{b'c''}{bc} \right] + \\ & \frac{s}{6(1+\rho)^3} \times \frac{a''-a'''}{a} \left[\frac{2b''c''}{bc} + \frac{2b'''c'''}{bc} + \frac{b'''c''}{bc} + \frac{b''c'''}{bc} \right] + \\ & \qquad \qquad \qquad \&c \qquad \qquad \qquad \&c \qquad \qquad \qquad \&c, \end{aligned}$$

which being continued to n terms will be the required value of the given sum. But, since the quantities $a-a'$, $a'-a''$, $a''-a'''$, &c are supposed to be equal and uniform (that is, equal to $\delta = \frac{a-a}{n}$) the sum of the first n terms of these several perpendicular series will

Observations on Mr. Morgan's Method of investigating this Problem.

§ 251. The motives, which induced me to notice (in page 195) the strange method which Mr. Morgan has adopted in summing up the several series arising from the investigation of the twenty-seventh problem, must be my apology here for again detaining the reader whilst I expose the equally diffuse and obscure manner which he has also adopted in summing up the several series arising from the investigation of this problem.

It will be seen, from an inspection of the series in page 207, that such series may be expressed in the following manner :

$$\begin{aligned} & \frac{s}{(1+\rho)} \left[\frac{(a-a')bc}{3abc} + \frac{(a-a')bc'}{6abc} + \frac{(a-a')bc''}{6abc} + \frac{(a-a')bc'''}{3abc} \right] + \\ & \frac{s}{(1+\rho)^2} \left[\frac{(a-a'')bc'}{3abc} + \frac{(a-a'')bc''}{6abc} + \frac{(a-a'')bc'''}{6abc} + \frac{(a-a'')bc''''}{3abc} \right] + \\ & \frac{s}{(1+\rho)^3} \left[\frac{(a-a''')bc''}{3abc} + \frac{(a-a''')bc'''}{6abc} + \frac{(a-a''')bc''''}{6abc} + \frac{(a-a''')bc'''''}{3abc} \right] + \\ & \qquad \qquad \qquad \&c \qquad \qquad \qquad \&c \qquad \qquad \qquad \&c; \end{aligned}$$

and this is the way in which Mr. Morgan has thought

be equal to $s \cdot \frac{a-a}{an}$ multiplied into $\frac{1+(BC)^{t-1}}{3(1+\rho)} + \frac{(BC)^t}{3} + \frac{1+(BC)^{t-1}}{6(1+\rho)} \cdot \frac{b}{b} + \frac{(BC)^t}{6} \cdot \frac{b}{b}$. If A is the youngest of the three lives, and differs much from the ages of both the other lives, this formula will be tolerably correct, even if the several perpendicular series are continued to the utmost extent of human life; and will consequently serve in such cases to find a *near* value of the sum for the whole continuance of the lives: but it must always be used with proper caution.

proper to represent the present value of the given sum, to be received on the contingency mentioned in the problem.*

* In this investigation I have adopted the same symbols which have been used throughout the whole of this work: but, lest I may be accused of having misrepresented Mr. Morgan, I shall here give his own solution, and in his own style.

Let r denote £1 increased by its interest for a year. Let a denote the number of persons living at the age of A; and a', a'', a''' , &c, the decrements of life at the end of the first, second, third, &c, years from the age of A. Let b denote the number of persons living at the age of B; and m, n, o, p , &c, the number living at the end of one, two, three, &c, years from that age. Let c denote the number of persons living at the age of C; and d, e, f , &c, the number living at the end of one, two, three, &c, years from that age. Let F denote the value of an annuity on a life one year younger than B; and β the number of persons living at that age. Let K denote the value of an annuity on a life one year younger than C; and x the number of persons living at that age. Let P denote the value of an annuity one year older than B: and T , the value of an annuity one year older than C. And let the given sum be denoted by S .

Then (in the words of Mr. Morgan, *Phil. Trans.* for 1791, p. 249) the whole value of the given sum will be = $\frac{S}{3abc} \times$

$$\left[\frac{abc}{r} + \frac{amd}{r^2} + \frac{aen}{r^3} + \&c \right] + \frac{S}{6abc} \times \left[\frac{amc}{r} + \frac{and}{r^2} + \frac{aoe}{r^3} + \&c \right] + \frac{S}{6abc} \times \left[\frac{adb}{r} + \frac{aem}{r^2} + \frac{afn}{r^3} + \&c \right] + \frac{S}{3abc} \times \left[\frac{amd}{r} + \frac{aen}{r^2} + \frac{afo}{r^3} + \&c \right].$$

But, the first of

these series (he observes) is = $\frac{S}{3} \times \frac{\beta x(FK - AFK)}{bc} - \frac{S}{3r} \times (BC - ABC)$; the second is = $\frac{S}{6} \times \frac{x(LK - ABK)}{c} - \frac{S}{6r} \times$

§ 252. In order to sum up these four collateral series he proceeds in the following manner. The

$$\begin{aligned} & \frac{m(PC-APC)}{b}; \text{ the third is } = \frac{S}{6} \times \frac{\beta(FC-AFC)}{b} - \frac{S}{6r} \times \\ & \frac{d(BT-ABT)}{c}; \text{ the fourth is } = \frac{S}{3} \times (BC-ABC) - \frac{S}{3r} \times \\ & \frac{md(PT-APT)}{bc}. \text{ Therefore these several expressions being ad-} \\ & \text{ded together will be found } = S \text{ into } \frac{x}{3c} \times \left[\frac{\beta(FK-AFK)}{b} + \right. \\ & \left. \frac{BK-ABK}{2} \right] + \frac{\beta}{6b} \times (FC-AFC) + \frac{r-1}{3r} \times (BC-ABC) - \\ & \frac{m(PC-APC)}{6br} - \frac{d}{3cr} \times \left[\frac{BT-ABT}{2} + \frac{m(PT-APT)}{b} \right]. \end{aligned}$$

Thus far Mr. Morgan.—And these several values, being expressed by the characters made use of in the present work, will be found to agree with the process given in the text. But it may be demonstrated that

$$\begin{aligned} \frac{\beta_x.FK}{bc} - \frac{BC}{r} &= \frac{1}{r} \\ \frac{x.BK}{c} - \frac{m.PC}{br} &= \frac{m}{br} \\ \frac{\beta.FC}{b} - \frac{d.BT}{cr} &= \frac{d}{cr} \\ BC - \frac{dm.PT}{bcr} &= \frac{dm}{bcr}: \end{aligned}$$

whence, the complex formula above given may be divested of all those expressions involving *two* joint lives, and may be more simply expressed by S multiplied into $\frac{1-(r-1)ABC}{3r} + \frac{1}{3bc} \times$
 $\left[\frac{(1+APT)dm}{r} - AFK.\beta_x \right] + \frac{1}{6b} \left[\frac{(1+APC)m}{r} - AFC.\beta \right] +$
 $\frac{1}{6c} \left[\frac{(1+ABT)d}{r} - ABK.x \right]$: which corresponds with the formula given in the text.

first of them he expands (as in *Phil. Trans.* for 1789, page 44) into the more complex series

$$\begin{aligned} & \frac{s}{(1+p)} \left[\frac{bc}{3bc} - \frac{abc}{3abc} \dots \dots \dots \right] + \\ & \frac{s}{(1+p)^2} \left[\frac{bc}{3bc} - \frac{abc}{3abc} - \frac{bc}{3bc} + \frac{abc}{3abc} \right] + \\ & \frac{s}{(1+p)^3} \left[\frac{bc}{3bc} - \frac{abc}{3abc} - \frac{bc}{3bc} + \frac{abc}{3abc} \right] + \\ & \qquad \qquad \qquad \&c \qquad \qquad \&c \qquad \&c, \end{aligned}$$

the sum of all which (independent of the common multiple $\frac{s}{3}$) he makes equal to $BC \times \frac{bc}{bc} - ABC \times \frac{bc}{bc} - \frac{BC}{(1+p)} + \frac{ABC}{(1+p)}$.

By proceeding in a similar manner, he makes the second collateral series, in page 214, (independent of the common multiple $\frac{s}{6}$) equal to $BC \times \frac{c}{c} - ABC \times \frac{c}{c} - \frac{BC}{(1+p)} \times \frac{b}{b} + \frac{ABC}{(1+p)} \times \frac{b}{b}$: the third equal to $BC \times \frac{b}{b} - ABC \times \frac{b}{b} - \frac{BC}{(1+p)} \times \frac{c}{c} + \frac{ABC}{(1+p)} \times \frac{c}{c}$: and the fourth equal to $BC - ABC - \frac{BC}{(1+p)} \times \frac{bc}{bc} + \frac{ABC}{(1+p)} \times \frac{bc}{bc}$. Consequently, the sum of the four collateral series in page 214 (or the total present value of the given sum to be received on the contingency mentioned in the problem) will, as he observes, be found equal to s multiplied into $BC \times \frac{bc}{3bc} -$

$$\begin{aligned}
 & \frac{ABC}{1} \times \frac{\frac{b}{1}}{3bc} - \frac{BC}{3(1+\rho)} + \frac{ABC}{3(1+\rho)} + BC \times \frac{\frac{c}{1}}{6c} - \\
 & \frac{ABC}{1} \times \frac{\frac{c}{1}}{6c} - \frac{BC}{(1+\rho)} \times \frac{\frac{b}{1}}{6b} + \frac{ABC}{(1+\rho)} \times \frac{\frac{b}{1}}{6b} + BC \times \\
 & \frac{\frac{b}{1}}{6b} - \frac{ABC}{1} \times \frac{\frac{b}{1}}{6b} - \frac{BC}{(1+\rho)} \times \frac{\frac{c}{1}}{6c} + \frac{ABC}{(1+\rho)} \times \frac{\frac{c}{1}}{6c} + \\
 & \frac{BC}{3} - \frac{ABC}{3} - \frac{BC}{(1+\rho)} \times \frac{\frac{b}{1}}{3bc} + \frac{ABC}{(1+\rho)} \times \frac{\frac{b}{1}}{3bc}.
 \end{aligned}$$

§ 253. Now, a moment's attention to the steps of this process will convince us that all those quantities, in which *two* joint lives are involved, are *unnecessarily* introduced into the above formula; since they evidently destroy each other, and may consequently be made to vanish altogether. For,

$$\begin{aligned}
 BC \times \frac{\frac{b}{1}}{bc} - \frac{BC}{(1+\rho)} &= \frac{1}{(1+\rho)} * \\
 BC \times \frac{\frac{c}{1}}{c} - \frac{BC}{(1+\rho)} \times \frac{\frac{b}{1}}{b} &= \frac{\frac{b}{1}}{b(1+\rho)} \\
 BC \times \frac{\frac{b}{1}}{b} - \frac{BC}{(1+\rho)} \times \frac{\frac{c}{1}}{c} &= \frac{\frac{c}{1}}{c(1+\rho)} \\
 BC - \frac{BC}{(1+\rho)} \times \frac{\frac{b}{1}}{bc} &= \frac{\frac{b}{1}}{bc(1+\rho)};
 \end{aligned}$$

whence, the complex formula given by Mr. Morgan may be reduced to *s* multiplied into $\frac{1-\rho ABC}{3(1+\rho)}$

* It is evident, *on inspection*, that this is the value of the first and third collateral series in page 217; because all the subsequent terms *after the first term* manifestly destroy each other. Consequently the sum of those two collateral series will be equal to $\frac{1-\rho}{3}$

$\frac{1}{3bc} \left[\frac{(1+ABC)bc}{(1+\rho)} - ABC.bc \right] + \frac{1}{6b} \left[\frac{(1+ABC)b}{(1+\rho)} - \right.$
 $\left. ABC.b \right] + \frac{1}{6c} \left[\frac{(1+ABC)c}{(1+\rho)} - ABC.c \right]$; and which
 is the very same as that which I have given in page

first term only: that is, (independent of the common multiple $\frac{s}{3}$) equal to $\frac{1}{(1+\rho)} \times \frac{bc}{bc} = \frac{1}{(1+\rho)}$. In like manner may be demonstrated the truth of the remaining equations.

I shall however make another observation here, which I think of considerable importance. Although these equations are strictly and mathematically correct, yet (owing to the imperfection of the Tables which show the value of Life Annuities) they oftentimes vary in a material degree when solved *arithmetically*: an inaccuracy which, being repeated so frequently in the same formula, must render the solution, according to Mr. Morgan's investigation, extremely erroneous.

Thus, the compound quantity $BC \cdot \frac{bc}{bc} - \frac{BC}{(1+\rho)}$ is equal to $\frac{1}{(1+\rho)}$, and (if simplicity and elegance were out of the question, and no regard were paid to the labour and intricacy of the solution) might be substituted instead. Let us, however, reduce these expressions to *numbers*: for which purpose let us take the age of B 15 years, the age of C 18 years, the rate of interest 4 per cent, and the probabilities of living as at Northampton. In this case we should have $\frac{1}{(1+\rho)} = .962$; but the value of $BC \cdot \frac{bc}{bc} - \frac{BC}{(1+\rho)}$ is equal to $13.566 - 12.614 = .952$ only: a difference which arises from the want of more enlarged tables of the value of annuities on two joint lives, and which will have a considerable effect upon the proper solution of the problem. This remark extends to *all* the formulæ inserted by Mr. Morgan in the several volumes of the *Philosophical Transactions*; and will show that those formulæ, however correct in principle, are of little or no use in *practice*.

211, except as to the substitution of the second term

$$+ \frac{1}{3bc} \left[\frac{(1+ABC)bc}{(1+\rho)} - ABC.bc \right]$$
 for its equal —

$$\frac{1}{3a} \left[\frac{(1+ABC)a}{(1+\rho)} - ABC.a \right],$$
 agreeably to what I have
 said in the note in page 210.

From this it will be readily seen that Mr. Morgan's formula consists of *double the number of terms* that are requisite for solving the problem; whereby it is rendered not only unnecessarily intricate and laborious, but also liable to considerable error in the *arithmetical* solution. Indeed it is manifest, from the nature of the series in page 207, that no expression involving *two* joint lives can properly arise: and it is singular that this consideration did not induce Mr. Morgan (if his object were perspicuity and information) to retrace the steps of his investigation, in order to render the process at once more simple and clear.

§ 254. But, there is another part of Mr. Morgan's investigation on which I think it equally necessary to make some observations. He states that the theorem which he has deduced, as above, "gives the exact value when either B or C is the oldest of the three lives; but when A is the oldest, it will be *necessary**

* It certainly is not *necessary* to change the symbols; since, in whatever manner the characters are varied, we must come to the same general series at last: neither does such a measure facilitate the means of investigation. Indeed I am at a loss to conceive why Mr. Morgan should have attempted to render so obscure this interesting branch of the subject. As a display of ingenuity it must completely fail: since, his papers will be read with very different emotions from those of admiration.

“to change the symbols &c:” and he has deduced *another* formula, for the solution of the problem in this case, involving *other* characters and symbols which render the investigation still more confused. From all which he would seem to imply that the first formula was not applicable to the second case.

But the second formula which he has deduced is still the *same* (mixed however with the useless and extraneous quantities above alluded to) as that which I have given in page 211; except as to the exchange of the last term $+ \frac{1}{\delta c} \left[\frac{(1+ABC)'c}{(1+i)} - ABC_{c,c} \right]$ for its equal $- \frac{1}{\delta ab} \left[\frac{(1+ABC)'ab}{(1+i)} - ABC_{ab,ab} \right]$, agreeably to what I have said in the note in page 210.* And it may

* Mr. Morgan's formula, expressed by his own ambiguous characters, is in this case equal to S multiplied into $\frac{\beta}{3b} \times \left[\frac{\alpha(HF + \frac{1}{2}HFC)}{a} - (AF + \frac{1}{2}AFC) \right] + \frac{1}{6} \times \left[\frac{\alpha(HB + 2HBC)}{a} - (AB + 2ABC) \right] + \frac{1}{3r} \times \left[\frac{s(BN - BNC)}{a} - (AB - ABC) \right] + \frac{m}{\delta br} \times \left[\frac{s(PN - PNC)}{a} - (AP - APC) \right]$. Where H and N denote the value of an annuity on a life one year younger and one year older than A respectively; and where a and s denote the number of persons living at those ages respectively: the other symbols being the same as in the note in page 215.

But this complex formula, being divested of those *useless* quantities which express the values of annuities on *two* joint lives, may be reduced to the simple terms alluded to in the text. See the original formula in *Phil. Trans.* for 1791, page 251: or in Price's *Obs. on Rev. Pay.* Problem I in note (P).

be here useful to remark that the formula, which I have given in page 211, is *universally* true, and does not depend on the seniority of the lives concerned. The substitutions above mentioned are mere arbitrary arrangements for the numerical solution of the problem; which may be occasionally used, or not, at the option of the calculator.

§ 255. It is easy to perceive, from what is here said, that the formula (denoting the value of the sum depending on the contingency in this problem) may be varied in *three* different ways, merely by the manner of summing up the several collateral series given in page 207; agreeably to the method laid down in the note in page 209. But such variation is by no means necessary or useful. Mr. Morgan has, in a very circuitous and diffuse manner, demonstrated only *two* of these ways:* but, by this additional attempt at elucidation, he has certainly rendered the subject still more confused. The introduction, indeed, of *unnecessary quantities* into any investigation (but more particularly the retaining of them in any resulting formulæ) and the capricious changing of the symbols employed, ought to be universally reprobated; not only as subversive of the true ends of science (whose object is

* That is, when B or C is the oldest of the three lives, and when A is the oldest of the three lives. But, in none of the problems (inserted by him in the *Phil. Trans.* for 1788, 1789, or 1791) do the values depend on the *seniority* of the lives concerned: for, *either* of the formulæ deduced by him, in the respective problems, will be *equally* correct whether A, B, or C be the oldest life.

information and not mystery), but also as destructive of all good taste in mathematical reasoning.

I have thought it proper to make these observations in this place, because the present problem is of considerable importance in enabling us to determine the value of many of the subsequent problems; and is made use of by him for that purpose: therefore the remarks here made will equally apply to those problems in which Mr. Morgan has so used it. Indeed, I believe *there is not a single problem* inserted by him in any of his papers in the *Philosophical Transactions*, respecting the value of Contingent Assurances, wherein this prolix and confused method has not been adopted, in order to determine the same.

PROBLEM XXX.*

§ 256. To determine the present value of a given sum payable on the decease of A, provided he be the *second* that fails of three given lives A, B, C.

SOLUTION.

The sum may be received at the end of the first year, on the happening of either of three different events: 1° that all the three lives fail in that year, A having died second: 2° that A and B fail in the year,

* Morgan, Prob. 20; and in Phil. Trans. for 1791, Prob. 2, page 253.

A having died last, and that C lives to the end of it :
 3° that A and C fail in the year, A having died last,
 and that B lives to the end of it. The probabilities of
 the happening of these several events are respectively
 $\frac{(a-a')(b-b')(c-c')}{3abc}$, $\frac{(a-a')(b-b')c'}{2abc}$ and $\frac{(a-a')(c-c')b'}{2abc}$;
 which, being added together and multiplied by
 $s(1+p)^{-1}$, will give the expectation of receiving the
 sum at the end of the first year.

But in the second and following years, the given
 sum may be received on the happening of either of
 seven different events: 1° that all the three lives be-
 come extinct in the year, A having died second: 2°
 that A and B both fail in the year, A having died
 last, and that C lives to the end of it: 3° that A and
 C both fail in the year, A having died last, and that
 B lives to the end of it: 4° that A and B both fail
 in the year, B having died last, and C having failed
 in either of the preceding years: 5° that A and C
 both fail in the year, C having died last, and B having
 failed in either of the preceding years: 6° that only
 A dies in the year, B living to the end of it, and C
 having died in either of the preceding years: 7° that
 only A dies in the year, C living to the end of it, and
 B having died in either of the preceding years. The
 probabilities of the happening of these several events
 in the second year, are respectively $\frac{(a-a'')(b-b'')(c-c'')}{3abc}$,
 $\frac{(a-a'')(b-b'')c''}{2abc}$, $\frac{(a-a'')(c-c'')b''}{2abc}$, $\frac{(a-a'')(b-b'')}{2ab}(1-\frac{c}{c'})$,

$\frac{(a'-a'')(c'-c'')}{2ac} \left(1 - \frac{b'}{b}\right), \quad \frac{(a'-a'')b'}{ab} \left(1 - \frac{c'}{c}\right), \text{ and }$
 $\frac{(a'-a'')c''}{ac} \left(1 - \frac{b''}{b}\right)$: which, being added together and multiplied by $s(1+\rho)^{-2}$, will give the expectation of receiving the sum at the end of the second year. In like manner it will be found that the probabilities of the happening of these several events in the third year will be respectively denoted by $\frac{(a''-a''')(b''-b''')(c''-c''')}{3abc}$, $\frac{(a''-a''')(b''-b''')c'''}{2abc}$, $\frac{(a''-a''')(c''-c''')b'''}{2abc}$, $\frac{(a''-a''')(b''-b''')}{2ab} \left(1 - \frac{c'''}{c}\right)$, $\frac{(a''-a''')(c''-c''')}{2ac} \left(1 - \frac{b'''}{b}\right)$, $\frac{(a''-a''')b'''}{ab} \left(1 - \frac{c'''}{c}\right)$, and $\frac{(a''-a''')c'''}{ac} \times \left(1 - \frac{b'''}{b}\right)$: which, being added together and multiplied by $s(1+\rho)^{-3}$, will give the expectation of receiving the sum at the end of the third year: and so on for every subsequent year to the utmost extent of human life: the sum of all which yearly expectations will be the total present value of the given sum to be received on the above contingency.

§ 257. Now if these several annual expectations be reduced to their least terms, and then arranged under each other as in the preceding problem, they will be found to form sixteen collateral series; the sum of all which will be equal to s multiplied into

$$\frac{1-\rho AB}{2(1+\rho)} - \frac{1+A'B}{2(1+\rho)} \cdot \frac{a}{a} + A'B \cdot \frac{a}{2a} + \frac{1-\rho AC}{2(1+\rho)} - \frac{1+A'C}{2(1+\rho)} \cdot \frac{a}{a}$$

$$\begin{aligned}
 & + AC \cdot \frac{a}{2a} - \frac{2}{3} \times \frac{1-\rho ABC}{(1+\rho)} + \frac{1+ABC}{3(1+\rho)} \cdot \frac{2a}{a} - ABC \cdot \frac{2a}{3a} \\
 & - \frac{1+ABC}{3(1+\rho)} \cdot \frac{b}{b} + ABC \cdot \frac{b}{3b} - \frac{1+ABC}{3(1+\rho)} \cdot \frac{c}{c} + ABC \cdot \frac{c}{3c}.
 \end{aligned}$$

But, the first three terms here given are, by Prob. XXVII, equal to A^B ; the next three terms are, by the second corollary to the same problem, equal to A^C ; and the remaining seven terms are, by Prob. XXIX, equal to $-2A^{BC}$. Consequently, the total present value of the given sum depending on this contingency will be equal to $s[A^B + A^C - 2A^{BC}]$.*

COROLLARY.

§ 258. If the three lives are equal, or of the same age A, then A^B and A^C will each of them become (as in Prob. XXVII, cor. 5) equal to $\frac{1-\rho AA}{2(1+\rho)}$; and A^{BC} will become (as in Prob. XXIX, cor. 2) equal to $\frac{1-\rho AAA}{3(1+\rho)}$. Consequently the present value of the

* It will be evident from an inspection of the several quantities, whence this formula is deduced, that a more convenient one might have been formed for the *numerical* solution of the problem: similar to the method pursued in the 39th and subsequent problems. But the present simple formula, at the same time that it is more easily retained in the memory, has also the additional advantage of enabling us readily to discover any error which may arise in the process. I have therefore on this account inserted it: but the reader may, from the values of the several series above given, arrange the formula in such other manner as he may find most convenient to himself. The same observations will apply to many of the subsequent problems in this chapter.

given sum will in such case be represented by $s \times \frac{1 - \rho(3AA - 2AAA)}{3(1 + \rho)}$: or, by one third of the present value of the given sum payable on the extinction of any two out of the three given lives.

PROBLEM XXXI.*

§ 259. To determine the present value of a given sum payable on the decease of A, provided he be the last which fails of three given lives A, B, C.

SOLUTION.

It is evident that the sum can be received at the end of the first year, only on the extinction of all the lives, A having died last, the probability of which is $\frac{(a-a') \times (b-b') \times (c-c')}{3abc}$; and which, being multiplied by $s(1 + \rho)^{-1}$, will give the expectation of receiving the sum at the end of the first year.

But in the second and following years the sum may be received on the happening of either of four different events: 1° that all the lives fail in the year, A having died last: 2° that A and B fail in the year, A having died last, and that C fails in either of the preceding years: 3° that A and C fail in the year, A having died last, and that B fails in either of the preceding years: 4° that only A dies in the year, B

* Morgan, Prob. 21; and in Phil. Trans. for 1791, Prob. 3, page 256. Simpson, Prob. 30.

and C having failed in either of the preceding years. The probabilities of these several events happening in the second year are respectively $\frac{(a'-a'')(b'-b'')(c'-c'')}{3abc}$, $\frac{(a'-a'')(b'-b'')}{2ab} \left(1 - \frac{c'}{c}\right)$, $\frac{(a'-a'')(c'-c'')}{2ac} \left(1 - \frac{b'}{b}\right)$, and $\frac{a'-a''}{a} \left(1 - \frac{b'}{b}\right) \times \left(1 - \frac{c'}{c}\right)$: which being added together and multiplied by $s(1+\rho)^{-2}$, will give the expectation of receiving the sum at the end of the second year.

In like manner may be found the expectation of receiving the sum at the end of the third and every subsequent year to the utmost extent of human life: and, if these several yearly expectations be reduced to their lowest terms and arranged under each other, they will form eighteen collateral series, the sum of all which will be the present value required: and which will be found to be equal to s multiplied into $\frac{1-\rho A}{(1+\rho)}$ —
 $\frac{1-\rho AB}{2(1+\rho)} + \frac{1+\rho AB}{2(1+\rho)} \cdot \frac{a'}{a} - AB \cdot \frac{a'}{2a} - \frac{1-\rho AC}{2(1+\rho)} + \frac{1+\rho AC}{2(1+\rho)} \cdot \frac{a'}{a}$
 $- AC \cdot \frac{a'}{2a} + \frac{1-\rho ABC}{3(1+\rho)} - \frac{1+\rho ABC}{3(1+\rho)} \cdot \frac{a'}{a} + ABC \cdot \frac{a'}{3a}$
 $\frac{1+\rho ABC}{6(1+\rho)} \cdot \frac{b'}{b} - ABC \cdot \frac{b'}{6b} + \frac{1+\rho ABC}{6(1+\rho)} \cdot \frac{c'}{c} - ABC \cdot \frac{c'}{6c}$. But
 this expression is equal to $s \left[\frac{1-\rho A}{(1+\rho)} - A^B - A^C + A^{BC} \right]$: which therefore denotes the present value required.

COROLLARY.

§ 260. If the three lives are equal, or of the same

age A, the present value of the given sum will, agreeably to what has been said in the corollary to the last problem, be denoted by $s \times \frac{1-\rho(3A-3AA+AAA)}{3(1+\rho)}$: or, by one third of the present value of the given sum to be received on the extinction of the longest of the three lives.

SCHOLIUM.

§ 261. If the present values of the given sum, as found by Prob. XXIX, XXX, and XXXI, be added together, they will be found equal to the present value of the same sum to be received on the decease of A; that is, the sum of those three values will be equal to $s \times \frac{1-\rho A}{(1+\rho)}$: which proves the accuracy of the investigations.

PROBLEM XXXII.*

§ 262. To determine the present value of a given sum payable on the decease of A, provided he be the *first or second* that fails of three given lives A, B, C.

SOLUTION.

The given sum may be received at the end of the first year, on the happening of either of four different

* Morgan, Prob. 24; and in Phil. Trans. for 1791, Prob. 5, page 261. Simpson's *Sup.* Prob. 40.

events: 1° that all the three lives fail in the year, A having died first or second: 2° that A and B both die in the year, and that C lives to the end of it; 3° that A and C both die in the year and that B lives to the end of it: 4° that only A dies in the year, and that B and C both live to the end of it. The probabilities of the happening of these several events are respectively denoted by $\frac{2(a-a')(b-b')(c-c')}{3abc}$, $\frac{(a-a')(b-b')c'}{abc}$, $\frac{(a-a')(c-c')b'}{abc}$, and $\frac{(a-a')b'c'}{abc}$: and which being multiplied by $s(1+p)^{-1}$, will give the expectation of receiving the sum at the end of the first year.

But, in the second and following years, the given sum may be received on the happening of either of eight different events: 1° that all the three lives fail in the year, A having died first or second: 2° that A and B both die in the year, and that C lives to the end of it: 3° that A and C both die in the year and that B lives to the end of it: 4° that only A dies in the year and that B and C both live to the end of it: 5° that A dies before B in the year, C having failed in either of the preceding years: 6° that A dies before C in the year, B having failed in either of the preceding years: 7° that only A dies in the year, C having died in either of the preceding years, and B living to the end of it: 8° that only A dies in the year, B having died in either of the preceding years, and C living to the end of it. The probabilities of the happening of these several events in the second

year are denoted respectively by $\frac{2(a-a')(b-b')(c-c')}{3abc}$,
 $\frac{(a-a')(b-b')''c}{abc}$, $\frac{(a-a')(c-c')''b}{abc}$, $\frac{(a-a')''b''c}{abc}$, $\frac{(a-a')(b-b')}{2ab} \times$
 $(1 - \frac{c'}{c})$, $\frac{(a-a')(c-c')}{2ac}(1 - \frac{b'}{b})$, $\frac{(a-a')''b''}{ab}(1 - \frac{c'}{c})$, and
 $\frac{(a-a')''c''}{ac}(1 - \frac{b'}{b})$: which being added together, and
multiplied by $s(1+e)^{-2}$, will give the expectation of
receiving the sum at the end of the second year. In
like manner we may find the expectation of receiving
the sum at the end of the third and every succeeding
year to the utmost extent of human life: the sum of
all which expectations will be the total present value
of the given sum, to be received on the above con-
tingency.

These several yearly expectations, being reduced to
their least terms and arranged under each other, will
form sixteen collateral series;* the sum of all which
is the present value required: and will be found equal
to $s[A^B + A^C - A^{BC}]$.†

* It would be needless for me to swell the present work by in-
serting the *several* values of these series, since (from the examples
which have preceded) they may be readily discovered by an in-
spection of the resulting formula. Few persons, perhaps, who
read this part of the treatise, will be satisfied without going through
the *several* steps of the process: in which case, such a measure
would not be requisite. And I cannot here avoid the opportunity
of recommending the reader to adopt this plan through all the
subsequent problems; as he will find it an excellent *praxis* for
this species of analysis.

† It may be here useful to remark that this formula may be ob-

COROLLARY.

§ 263. When the three lives are equal, or all of the same age A , this expression (agreeably to what has been said in the corollary to Prob. XXX) will become equal to $s \times \frac{2-\rho(3AA-AAA)}{3(1+\rho)}$: or, equal to the difference between the value of an assurance of the given sum on the two joint lives, and one third of the value of an assurance of the same sum on the three joint lives.

PROBLEM XXXIII.*

§ 264. To determine the present value of a given sum payable on the decease of A , provided he be the *second or third* that fails of three given lives A, B, C .

SOLUTION.

The given sum may be received at the end of the

tained by taking the *sum* of the values deduced from Prob. XXI and XXX; or, by taking the *difference* between $s \times \frac{1-\rho A}{(1+\rho)}$ and the value deduced from Prob. XXXI: which is evident from nature of the questions. For, the sum is to be received on death of A provided he dies *first or second*: or (which is the thing) it is to be received on the death of A provided he does *not* die *third*.

* Morgan, Prob. 25; and in Phil. Trans. for 1791, P

first year, on the happening of either of three different events: 1° that all the three lives become extinct in the year, A having died the second or third: 2° that A and B both fail in the year, A having died last, and that C lives to the end of it: 3° that A and C both fail in the year, A having died last, and that B lives to the end of it. The probabilities of the happening of these several events in the first year, are respectively $\frac{2(a-a')(b-b')(c-c')}{3abc}$, $\frac{(a-a')(b-b')c}{2abc}$, and $\frac{(a-a')(c-c')b}{2abc}$; which being added together and multiplied by $s(1+p)^{-1}$, will give the expectation of receiving the sum at the end of the first year.

But in the second and following years, the sum may be received on the happening of either of eight different events: 1° that all the three lives fail in the year, A having died second or third: 2° that A and B both fail in the year, A having died last, and that C lives to the end of it: 3° that A and C both fail in the year, A having died last, and that B lives to the end of it: 4° that A and B both die in the year, C having died in either of preceding years: 5° that A and C both die in the year, B having died in either of the preceding years: 6° that only A dies in the year, B living to the end of it, and C having died in either of the preceding years: 7° that only A dies in the year, C living to the end of it, and B having died in either of the preceding years: 8° that only A dies in the year, B and C having both failed in either of the preceding years. The probabilities of the hap-

pening of these several events in the second year are respectively $\frac{2(a'-a'')(b'-b'')(c'-c'')}{3abc}$, $\frac{(a'-a'')(b'-b'')c''}{2abc}$, $\frac{(a'-a'')(c'-c'')b''}{2abc}$, $\frac{(a'-a'')(b'-b'')}{ab}(1-\frac{c'}{c})$, $\frac{(a'-a'')(c'-c'')}{ac} \times (1-\frac{b'}{b})$, $\frac{(a'-a'')b''}{ab}(1-\frac{c'}{c})$, $\frac{(a'-a'')c''}{ac}(1-\frac{b'}{b})$, and $\frac{(a'-a'')}{a}(1-\frac{b'}{b}) \cdot (1-\frac{c'}{c})$: which, being added together and multiplied by $s(1+\rho)^{-2}$, will give the expectation of receiving the sum at the end of the second year. In like manner we may find the expectation of receiving the sum at the end of the third and every subsequent year to the utmost extent of human life: the sum of all which expectations will be the total present value of the given sum, depending on the above contingency.

These several yearly expectations, being reduced to their least terms and arranged under each other, will form ten collateral series: the sum of all which will be found equal to $s \left[\frac{1-\rho^A}{(1+\rho)} - A^{BC} \right]$.*

COROLLARY.

§ 265. When all the lives are equal, or of the

* The value of the sum depending on the contingency mentioned in this problem is manifestly equal to the *difference* between the value of such sum to be received on the decease of A, and its value depending on the contingency that A shall be the *first that* fails: or, it is equal to the *sum* of the two values found by Prob. XXX and XXXI.

same age A , the last term in this formula becomes equal to $\frac{1-\rho AAA}{3(1+\rho)}$ (agreeably to what has been already said in Prob. XXIX, cor. 2), consequently the whole expression is in this case reduced to $s \times \frac{2-\rho(3A-AAA)}{3(1+\rho)}$: that is, equal to the difference between the value of an assurance of the given sum on a single life, and one third of the value of an assurance of the same sum on the three joint lives.

PROBLEM XXXIV.*

§ 266. To determine the present value of a given sum payable on the decease of A , provided he be the *first or last* that fails of three given lives A, B, C .

SOLUTION.

The given sum may be received at the end of the first year on the happening of either of four different events: 1° that all the three lives fail in the year, A having died first or last: 2° that A and B both fail in the year, A having died first, and that C lives to the end of it: 3° that A and C both fail in the year, A having died first, and that B lives to the end of it: 4° that only A dies in the year, and that B and C both live to the end of it. The probabilities of the happening of these several events are respectively

* Morgau, Prob. 26; and in Phil. Trans. for 1791, Prob. 7, page 265.

$\frac{2(a-a')(b-b')(c-c')}{3abc}$, $\frac{(a-a')(b-b')c'}{2abc}$, $\frac{(a-a')(c-c')b'}{2abc}$, and $\frac{(a-a')b'c'}{abc}$: which, being added together and multiplied by $s(1+p)^{-1}$, will give the expectation of receiving the sum at the end of the first year.

But at the end of the second and following years the given sum may be received on the happening of either of seven different events: 1° that all the lives fail in the year, A having died first or last: 2° that A and B both fail in the year, A having died first, and that C lives to the end of it: 3° that A and C both fail in the year, A having died first, and that B lives to the end of it: 4° that only A dies in the year and that B and C both live to the end of it: 5° that A and B both fail in the year, A having died last, and C having failed in either of the preceding years: 6° that A and C both fail in the year, A having died last, and B having failed in either of the preceding years: 7° that only A dies in the year, B and C having both failed in either of the preceding years. The probabilities of the happening of these several events in the second year, are respectively $\frac{2(a-a'')(b-b'')(c-c'')}{3abc}$,

$$\frac{(a-a'')(b-b'')c''}{2abc}, \frac{(a-a'')(c-c'')b''}{2abc}, \frac{(a-a'')b''c''}{abc}, \frac{(a-a'')(b-b'')}{2ab} \times$$

$$\left(1 - \frac{c'}{c}\right), \frac{(a-a'')(c-c'')}{2ac} \left(1 - \frac{b'}{b}\right), \text{ and } \frac{a-a''}{a} \left(1 - \frac{b'}{b}\right) \times$$

$\left(1 - \frac{c'}{c}\right)$: which, being added together and multiplied by $s(1+p)^{-2}$, will give the expectation of receiving

the sum at the end of the second year. In like manner we may find the expectation of receiving the sum at the end of the third and every subsequent year to the utmost extent of human life: the sum of all which yearly values will be the total present value of the given sum to be received on the above contingency.

These several yearly expectations, being reduced to their least terms and arranged under each other, will form eighteen collateral series: the sum of all which will be found equal to $s \times \left[\frac{1-\rho A}{(1+\rho)} - A^B - A^C + 2A^{BC} \right]$.*

COROLLARY.

§ 267. When the three lives are equal, or of the same age A , this expression (agreeably to what has been said in the corollary to Prob. XXX) will become equal to $s \times \frac{2-\rho(3A-3AA+2AAA)}{3(1+\rho)}$: that is, equal to the difference between the value of an assurance of the given sum on a single life and on two joint lives, added to two thirds of the value of an assurance of the same sum on the three joint lives.

SCHOLIUM.

§ 268. If the present values of the given sum, as

* This formula may be obtained by taking the *sum* of the values deduced from Prob. XXIX and XXXI: or, by taking the *difference* between $s \times \frac{1-\rho A}{(1+\rho)}$ and the value deduced from Prob. XXX.

found by Prob. XXXII, XXXIII, and XXXIV, be added together, they will be found equal to twice the present value of the same sum to be received on the decease of A : that is, the sum of those three values will be equal to $2s \times \frac{1-\rho^A}{(1+\rho)}$.

PROBLEM XXXV.*

§ 269. To determine the present value of a given sum payable on the decease of A or B, provided *either* of them be the *first* that fails of three given lives A, B, C.

SOLUTION.

The given sum may be received at the end of any one year, on the happening of either of six different events: 1° that all the lives fail in the year, A or B having died first: 2° that A and B both die in the year, and that C lives: 3° that A and C both fail in the year, A having died first, and that B lives to the end of it: 4° that B and C both fail in the year, B having died first, and that A lives to the end of it: 5° that only A dies in the year, and that B and C both live to the end of it: 6° that only B dies in the year, and that A and C both live to the end of it. The probabilities of the happening of which several events in the first year are respectively

* Phil. Trans. for 1791, Prob. 8, page 267.

$\frac{2(a-a')(b-b')(c-c')}{3abc}$, $\frac{(a-a')(b-b')c'}{abc}$, $\frac{(a-a')(c-c')b'}{2abc}$,
 $\frac{(b-b')(c-c')a'}{2abc}$, $\frac{(a-a')b'b'}{abc}$, and $\frac{(b-b')a'a'}{abc}$: which, being
 added together and multiplied by $s(1+p)^{-1}$, will give
 the expectation of receiving the sum at the end of the
 first year.

In like manner may be found the expectation of re-
 ceiving the sum at the end of the second, third and
 every subsequent year to the utmost extent of human
 life: which several yearly values, reduced to their least
 terms and being arranged under each other as in Prob.
 XXIX, will form the following series,

$$\begin{aligned} & \left[\frac{4abc}{abc} - \frac{4a'b'c'}{abc} - \frac{a'bc}{abc} + \frac{ab'c'}{abc} - \frac{a'bc}{abc} + \frac{a'b'c'}{abc} + \frac{2a'b'c'}{abc} - \frac{2a'b'c'}{abc} \right] \\ & \left[\frac{4a'b'c'}{abc} - \frac{4a'b'c'}{abc} - \frac{a'b'c'}{abc} + \frac{a'b'c'}{abc} - \frac{a'b'c'}{abc} + \frac{a'b'c'}{abc} + \frac{2a'b'c'}{abc} - \frac{2a'b'c'}{abc} \right] \\ & \left[\frac{4a'b'c'}{abc} - \frac{4a'b'c'}{abc} - \frac{a'b'c'}{abc} + \frac{a'b'c'}{abc} - \frac{a'b'c'}{abc} + \frac{a'b'c'}{abc} + \frac{2a'b'c'}{abc} - \frac{2a'b'c'}{abc} \right] \end{aligned}$$

the sum of all which will be the total present value
 required.

§ 270. But, if we compare this general series with
 the one inserted in page 207, it will be found that the
 collateral series, of which it is composed, are
 in part the same which are there
 but the difference between them is
 a sign prefixed to the
 corresponding terms
 and another

be readily obtained from the process there laid down. For, the first and second collateral series will be thus found equal to $\frac{2s}{3} \times \frac{1-\rho ABC}{(1+\rho)}$; the third and fourth, equal to $-\frac{s}{6a} \left[\frac{(1+ABC)a'}{(1+\rho)} - ABC.a' \right]$; the fifth and sixth, equal to $-\frac{s}{6b} \left[\frac{(1+ABC)b'}{(1+\rho)} - ABC.b' \right]$; and the seventh and eighth, equal to $\frac{s}{3c} \left[\frac{(1+ABC)c'}{(1+\rho)} - ABC.c' \right]$. Therefore the total value of the general series above given will be equal to s multiplied into $\frac{2}{3} \times \frac{1-\rho ABC}{(1+\rho)} - \frac{1}{6a} \left[\frac{(1+ABC)a'}{(1+\rho)} - ABC.a' \right] - \frac{1}{6b} \left[\frac{(1+ABC)b'}{(1+\rho)} - ABC.b' \right] + \frac{1}{3c} \left[\frac{(1+ABC)c'}{(1+\rho)} - ABC.c' \right]$.

As we shall have occasion to refer again to this formula in some of the following problems, it will be convenient to denote it by a more simple expression; therefore let it be represented by $\mathcal{A}^C$: that is, let $\mathcal{A}^C$ denote the present value of £1 to be received on the above contingency; consequently $s \times \mathcal{A}^C$ will denote the present value of the given sum under the same circumstances.*

* By comparing this formula with the second formula in Prob. XXIX, cor. 1, it will be seen that $\mathcal{A}^C = \frac{1+ABC}{(1+\rho)} - C^{AB}$.

COROLLARY.

§ 271. When the three lives are equal, or of the same age A , the last three terms in the above expression destroy each other; and the whole is then reduced to $\frac{2s}{3} \times \frac{1-\rho AAA}{(1+\rho)}$: or, to two thirds of the present value of the given sum payable on the extinction of the three joint lives.

Observations on Mr. Morgan's Method of investigating this Problem.

§ 272. I cannot omit the present opportunity of recalling the reader's attention once more to the singular and confused manner which Mr. Morgan has adopted, in this case also, for finding the value of the general series in page 239: and I am the more induced to do this, because it appears that, the further Mr. Morgan proceeds in his subject, the more disorder and irregularity seems to run through the whole of his investigations. I would previously, however, intreat the reader's patience whilst I conduct him through this mathematical labyrinth.

The sum of the several expectations of receiving the given sum, on the contingency mentioned in this problem, Mr. Morgan makes equal to the following series

$$\begin{aligned}
& \frac{s}{(1+\varrho)} \left[\frac{2(a-a')bc}{3abc} - \frac{(a-a')bc}{6abc} + \frac{(a-a')bc'}{3abc} + \frac{(a-a')bc''}{6abc} + \frac{(b-b')c}{2bc} + \frac{(b-b')c'}{2bc} - \frac{(a-a')bc}{2abc} + \frac{(a-a')bc'}{2abc} + \frac{(a-a')bc''}{2abc} \right] + \\
& \frac{s}{(1+\varrho)^2} \left[\frac{2(a-a')bc}{3abc} - \frac{(a-a')bc}{6abc} + \frac{(a-a')bc'}{3abc} + \frac{(a-a')bc''}{6abc} + \frac{(b-b')c}{2bc} + \frac{(b-b')c'}{2bc} - \frac{(a-a')bc}{2abc} + \frac{(a-a')bc'}{2abc} + \frac{(a-a')bc''}{2abc} \right] + \\
& \frac{s}{(1+\varrho)^3} \left[\frac{2(a-a')bc}{3abc} - \frac{(a-a')bc}{6abc} + \frac{(a-a')bc'}{3abc} + \frac{(a-a')bc''}{6abc} + \frac{(b-b')c}{2bc} + \frac{(b-b')c'}{2bc} - \frac{(a-a')bc}{2abc} + \frac{(a-a')bc'}{2abc} + \frac{(a-a')bc''}{2abc} \right] + \\
& \quad \&c \quad \&c \quad \&c \quad \&c \quad \&c;
\end{aligned}$$

the whole value of which, he says, is equal to s multiplied into $BC \times \frac{b}{6bc} - ABC \times \frac{b}{6bc} + BC \times$

$$\begin{aligned}
& \frac{c}{3c} - ABC \times \frac{c}{3c} - BC \times \frac{b}{6b} + ABC \times \frac{b}{6b} + \frac{2\varrho(BC-ABC)}{3(1+\varrho)} + \frac{BC}{(1+\varrho)} \times \frac{b}{6b} - \frac{BC'}{(1+\varrho)} \times \\
& \frac{c}{3c} + \frac{ABC}{(1+\varrho)} \times \frac{c}{3c} - \frac{BC}{(1+\varrho)} \times \frac{b}{6b} + \frac{ABC}{(1+\varrho)} \times \frac{b}{6b} + \frac{\varrho(C-BC)}{2(1+\varrho)} + C \times \frac{c}{2c} - BC \times \frac{c}{2c} - \frac{C'}{(1+\varrho)} \times \frac{c}{2c} + \\
& \frac{BC}{(1+\varrho)} \times \frac{c}{2c} *
\end{aligned}$$

* These last five terms are the value of Mr. Morgan's Σ , expressed in his own circuitous manner. The reader will, at first, perhaps hesitate in believing that the *whole* of the above complex expression may be denoted by only *seven* of the terms made use of by Mr. Morgan !!! See the proper formula in page 240.

§ 273. Now, the last *six* collateral series in the above general expression may be reduced to the *two* following ones: viz.

$$\begin{aligned} & \frac{s}{(1+p)} \left[\frac{(b-b')a'c}{2abc} + \frac{(b-b')a''c'}{2abc} \right] + \\ & \frac{s}{(1+p)^2} \left[\frac{(b-b'')a''c'}{2abc} + \frac{(b-b'')a'''c''}{2abc} \right] + \\ & \frac{s}{(1+p)^3} \left[\frac{(b-b''')a'''c''}{2abc} + \frac{(b-b''')a''''c'''}{2abc} \right] + \\ & \qquad \qquad \qquad \&c \qquad \qquad \&c \qquad \qquad \&c, \end{aligned}$$

in which case, that expression, instead of being composed of *twenty** different series, might have been reduced to *twelve*. Or, had the whole of these collateral series been reduced to their simplest terms, they would have formed the *very same* expression as that given in page 239; and which consists of only *eight* different terms!!! But, owing to this careless and diffuse mode of treating the subject, it follows that the formula, which denotes the value of the given sum depending on the contingency mentioned in this problem, contains (upon Mr. Morgan's method of solution) no less than twenty different terms:† and when it is considered

* Each of the ten collateral series in page 242, evidently forms *two* distinct series when expanded by multiplication; the sum of *both* which it is necessary to find.

† Lest I should be charged with misrepresentation I shall here insert Mr. Morgan's formula in his own characters: viz S into $\frac{x}{3c} \left[\frac{\beta(FK-AFK)}{2b} + (BK-ABK) \right] - \frac{\beta(FC-AFC)}{6b} + \frac{2(r-1).(BC-ABC)}{3r} + \frac{m(PC-APC)}{6br} - \frac{d}{3cr} \left[(BT-ABT) + \frac{2n(PT-APT)}{2b} \right] + \Sigma (\Sigma \text{ denoting the value of } S \text{ on the con-}$

that, in order to obtain those terms, it is necessary* to sum up between *thirty and forty* different series, the reader is equally struck with surprise and indignation; and is left in doubt of the *motive* which could have induced the author to pursue so anomalous a course.

§ 274. There is, however, still another part of Mr. Morgan's investigation on which I consider myself equally obliged to make some comment. He says that the formula (above alluded to) gives the exact value when C is the oldest of the three lives; "but if A be the oldest, *the symbols must be changed.*" and the value of the given sum he has, in this case, denoted by *another* formula; which (being introduced in a new dress, and in new characters) appears materially different from the former one.† It is, however, (when divested of all its useless and extraneous quantities) precisely the same as that which I have

tingency of C's surviving B). The other symbols being the same as already explained in the note in page 215. See the original formula in *Phil. Trans.* for 1791, page 267; or in Price's *Obs. on Rev. Pay.* Note (P), Problem VIII.

* I mean upon Mr. Morgan's principles, as explained in page 217.

† Take Mr. Morgan's own words. "The general rule expressing the value of the reversion will be = S into $\frac{(r-1).(V-AB)}{r}$

$$- \frac{a}{3a} \left[\frac{\beta(HF-HFC)}{b} + \frac{HB-HBC}{2} \right] - \frac{\beta(AF-AFC)}{6b} + \frac{2(r-1).(AB-ABC)}{3r} + \frac{m(AP-APC)}{6br} + \frac{s}{3ar} \left[\frac{(BN-BNC)}{2} + \right.$$

given in page 240, except as to the substitution of

$$- \frac{1}{3ab} \left[\frac{(1 + \overset{!}{ABC}) \overset{!}{ab}}{(1 + \ell)} - \underset{!}{ABC} \underset{!}{ab} \right] \text{ for its equal } +$$

$$\frac{1}{3c} \left[\frac{(1 + \overset{!}{ABC}) \overset{!}{c}}{(1 + \ell)} - \underset{!}{ABC} \underset{!}{c} \right]; \text{ agreeably to the principles explained in the note in page 210.}$$

§ 275. It surely cannot be unknown to Mr. Morgan that the value of the given sum does not in this case depend upon the *seniority* of the lives concerned.* For, let the ages of A, B, and C be what they may, and let him change his symbols as often as he pleases, he must still come to the *same* general series, at last, as that given in page 239: the sum of which may be expressed in three different ways, according to the method of summing up the several collateral series of which it is composed.† And *either* of these formulæ will denote the value of the given sum, whether A, B, or C be the oldest of the three lives.

$\frac{m(PN - PNC)}{b} \Big] \text{."}$ Where V denotes the perpetuity; and the remaining letters, the same quantities as already explained in the note in page 221. See the original formula in *Phil. Trans.* for 1791, page 268; and in Price's *Obs. on Rev. Pay.* Note (P), Problem VIII.

* This observation will apply to the whole of the Problems inserted by Mr. Morgan in the *Phil. Trans.* for 1788, 1789 and 1791. See the note in page 222.

† See what has been already said on this subject in page 222, and in the note to page 220.

These observations are the more necessary to be attended to, because the solution to the present problem is of use in enabling us to determine the value of sums depending on the contingencies mentioned in several of the subsequent problems. Consequently, the more simple we render the present formula, the more easily we may obtain the solution to those problems. This motive, therefore, will be a sufficient apology for the present digression.

PROBLEM XXXVI.*

§ 276. To determine the present value of a given sum payable on the decease of A or B, provided *either* of them be the *second* that fails of three given lives A, B, C,

SOLUTION,

The given sum may be received at the end of the first year, on the happening of either of four different events: 1° that all the three lives fail in that year, A or B having died second: 2° that A and B both die in that year, and C lives to the end of it: 3° that A and C both fail in that year, A having died last, and that B lives to the end of it: 4° that B and C both fail in that year, B having died last, and that A lives to

* Phil. Trans. for 1791, page 269.

the end of it. The probabilities of the happening of these several events are respectively $\frac{2(a-a')(b-b')(c-c')}{3abc}$, $\frac{(a-a')(b-b')c'}{abc}$, $\frac{(a-a')(c-c')b'}{2abc}$, and $\frac{(b-b')(c-c')a'}{2abc}$: which being added together, and multiplied by $s(1+e)^{-1}$, will give the expectation of receiving the sum at the end of the first year.

But at the end of the second and following years, the sum may be received on the happening of either of eleven different events: 1° that all the lives fail in the year, A or B having died second: 2° that A and B both die in the year, and that C lives to the end of it: 3° that A and C both fail in the year, A having died last, and that B lives to the end of it: 4° that B and C both fail in the year, B having died last, and that A lives to the end of it: 5° that A and B both die in the year, C having died in either of the preceding years: 6° that A and C both fail in the year, A having died last, and B having died in either of the preceding years: 7° that B and C both fail in the year, B having died last, and A having died in either of the preceding years: 8° that only A dies in the year, B living to the end of it, and C having died in either of the preceding years: 9° that only A dies in the year, C living to the end of it, and B having died in either of the preceding years: 10° that only B dies in the year, A living to the end of it, and C having died in either of the preceding years: 11° that

only B dies in the year, C living to the end of it, and A having died in either of the preceding years. The probabilities of the happening of these several events in the second year are respectively $\frac{2(a''-a')(b''-b')(c''-c')}{3abc}$, $\frac{(a''-a')(b''-b')c''}{abc}$, $\frac{(a''-a')(c''-c')b''}{2abc}$, $\frac{(b''-b')(c''-c')a''}{2abc}$, $\frac{(a''-a')(b''-b')}{ab} \times (1 - \frac{c'}{c})$, $\frac{(a''-a')(c''-c')}{2ac} (1 - \frac{b'}{b})$, $\frac{(b''-b')(c''-c')}{2bc} (1 - \frac{a'}{a})$, $\frac{(a''-a')b''}{ab} (1 - \frac{c'}{c})$, $\frac{(a''-a')c''}{ac} (1 - \frac{b'}{b})$, $\frac{(b''-b')a''}{ab} (1 - \frac{c'}{c})$, and $\frac{(b''-b')c''}{bc} (1 - \frac{a'}{a})$: which being added together, and multiplied by $s(1+\rho)^{-2}$, will give the expectation of receiving the sum at the end of the second year. In like manner may be found the expectation of receiving the sum at the end of the third and every subsequent year to the utmost extent of human life: the sum of all which yearly values will be the total present value of the given sum, to be received on the above contingency.

Now these several yearly expectations, being reduced to their lowest terms and arranged under each other, will form eighteen collateral series: the sum of all which will be found equal to $s \left[\frac{1-\rho AB}{(1+\rho)} + A^c + B^c - 2AB^c \right]$.

COROLLARY.

§ 277. When all the lives are equal, or of the

same age A , this expression (agreeably to what has been said in the corollary to Prob. XXX) will become equal to $\frac{2s}{3} \times \frac{1 - \rho(3AA - 2AAA)}{(1 + \rho)}$: or, to two thirds of the present value of the given sum payable on the extinction of any two out of three given lives.

PROBLEM XXXVII.*

§ 278. To determine the present value of a given sum payable on the decease of A or B , provided *either* of them be the *last* that fails of three given lives A , B , C .

SOLUTION.

The given sum can be received at the end of the first year, on the happening of one event only; viz. the extinction of all the lives in that year, C having been the first or second that failed. The probability of this event is $\frac{2(a-a')(b-b')(c-c')}{3abc}$; which being multiplied by $s(1 + \rho)^{-1}$, will give the expectation of receiving the sum at the end of the first year.

But, in the second and following years, the given sum may be received on the happening of either of six different events: 1° that all the three lives fail in the year, C having died first or second: 2° that only A and B fail in the year, C having died in either of the

* Phil. Trans. for 1791, Prob. 10, page 272.

preceding years: 3^o that A and C both fail in the year, A having died last, and B having died in either of the preceding years: 4^o that B and C both fail in the year, B having died last, and A having died in either of the preceding years: 5^o that only A dies in the year, B and C having both failed in either of the preceding years: 6^o that only B dies in the year, A and C having both failed in either of the preceding years. The probabilities of the happening of these several events in the second year are respectively

$$\frac{2(a'-a'')(b'-b'')(c'-c'')}{3abc}, \frac{(a'-a'')(b'-b'')}{ab} \left(1 - \frac{c'}{c}\right), \frac{(a'-a'')(c'-c'')}{2ac} \times$$

$$\left(1 - \frac{b'}{b}\right), \frac{(b'-b'')(c'-c'')}{2bc} \left(1 - \frac{a'}{a}\right), \frac{(a'-a'')}{a} \left(1 - \frac{b'}{b}\right) \times$$

$$\left(1 - \frac{c'}{c}\right), \text{ and } \frac{(b'-b'')}{b} \left(1 - \frac{a'}{a}\right) \cdot \left(1 - \frac{c'}{c}\right): \text{ which being}$$

added together and multiplied by $s(1+p)^{-2}$, will give the expectation of receiving the sum at the end of the second year. In like manner we may find the expectation of receiving the sum at the end of the third year. And so on for all the subsequent years to the utmost extent of human life; the sum of all which yearly values will be the total present value of the given sum, payable on the above contingency.

These several yearly expectations, being reduced to their lowest terms and arranged under each other, will form twenty-two collateral series: the sum of all which will be found equal to $s \left[\frac{1-p(A+B-AB)}{(1+p)} + A^c - B^c + B^c \right]$.

COROLLARY.

§ 279. When the three lives are equal, or of the same age A , this expression will become equal to $\frac{2s}{3} \times \frac{1-\ell(3A-3AA+AAA)}{(1+\ell)}$: or, two thirds of the present value of the given sum, payable on the extinction of the longest of the three lives.

SCHOLIUM.

§ 280. If the present values of the given sum, as found by Prob. XXXV, XXXVI, and XXXVII, be added together, they will be found equal to the present value of the same sum to be received on the decease of A , *added to* the present value of the same sum to be received on the decease of B : that is, the sum of such three values will be equal to $s \left[\frac{1-\ell A}{(1+\ell)} + \frac{1-\ell B}{(1+\ell)} \right]$.

PROBLEM XXXVIII.*

§ 281. To determine the present value of a given sum payable on the decease of A or B , provided *either* of them be the *first* or *second* that fails of three given lives A, B, C .

SOLUTION.

It is evident that in this case the payment of the given sum at the end of any one year depends wholly

* Phil. Trans. for 1800, Prob. 1, page 22.

on the extinction of the joint lives AB in that year, independent of C : its present value therefore will, by Prob. XXII, be in all cases equal to $s \times \frac{1 - {}_t p_{AB}}{(1 + i)}$.

SCHOLIUM.

§ 282. In the preceding problems I have deduced *correct* expressions for the value of reversionary sums depending on the several contingencies therein mentioned: but the remaining problems, for the most part, involve a contingency for which it is very difficult to find such an expression as will denote the true value of the same, and be likewise fit for general use. The contingency, to which I allude, is the probability that one life in particular will die before or after another, during any given period of their joint lives. This subject has been already discussed in the fifth chapter, where it is applied to the method of determining the present value of certain reversionary *annuities*: and I now come to consider it again in its application to the method of determining the present value of reversionary *sums*. In the investigation of the following problems, the contingencies above mentioned will be expressed in *italics*. And since, by means of the two Lemmata in Chapter V, we may obtain a more convenient expression for the expectations of receiving the given sum after the extinction of the oldest life involved, I shall divide the investigation into two distinct parts; the first of which will denote the value of

all the expectations for the first n years,* or during the continuance of the oldest life involved; and the second will denote the value of all the expectations after that period.

PROBLEM XXXIX.†

§ 283. To determine the present value of a given sum payable on the decease of A or B, provided *either* of them be the *second* or *third* that fails of three given lives A, B, C.

SOLUTION.

The given sum may be received at the end of the first year on the happening of either of four different events: 1° that all the lives fail in that year: 2° that A and B fail in that year, and that C lives to the end of it: 3° that A dies after C in that year, and that B lives to the end of it: 4° that B dies after C in that year, and that A lives to the end of it. The probabilities of the happening of these several events are respectively $\frac{(a-a')(b-b')(c-c')}{abc}$, $\frac{(a-a')(b-b')c}{abc}$, $\frac{(a-a')(c-c')b}{abc}$, and $\frac{(b-b')(c-c')a}{abc}$: which, being added together and multi-

* I shall here observe, once for all, that I take n to denote the number of years between the age of the oldest life and that age in the table of observations when human life becomes extinct: consequently the value of n will vary in each problem according to the seniority of the lives concerned.

† Phil. Trans. for 1806, Prob. 2, page 23.

plied by $s(1+p)^{-1}$, will give the expectation of receiving the sum at the end of the first year.

But in the second and following years, during the existence of the oldest life, the given sum may be received on the happening of either of thirteen different events: 1° that all the lives fail in the year: 2° that A and B fail in the year, and that C lives to the end of it: 3° that A and C both fail in the year, A having died last, and that B lives to the end of it: 4° that B and C both fail in the year, B having died last, and that A lives to the end of it: 5° that A and B die in the year, C having failed in either of the preceding years: 6° that A and C die in the year, B having failed in either of the preceding years: 7° that B and C die in the year, A having failed in either of the preceding years: 8° that only A dies in the year, B living to the end of it, and C having died in either of the preceding years: 9° that only A dies in the year, C living to the end of it, and B having died in either of the preceding years: 10° that only B dies in the year, A living to the end of it, and C having died in either of the preceding years: 11° that only B dies in the year, C living to the end of it, and A having died in either of the preceding years: 12° that only A dies in the year, and that B and C both fail in either of the preceding years, B having died first: 13° that only B dies in the year, and that A and C both fail in either of the preceding years, A having died first. The probabilities of the happening of these several events

in the second year, are respectively $\frac{(a-a'')(b-b'')(c-c'')}{abc}$,
 $\frac{(a-a'')(b-b'')c}{abc}$, $\frac{(a-a'')(c-c'')b}{2abc}$, $\frac{(b-b'')(c-c'')a}{2abc}$, $\frac{(a-a'')(b-b'')}{ab} \times$
 $(1 - \frac{c}{c})$, $\frac{(a-a'')(c-c'')}{ac}(1 - \frac{b}{b})$, $\frac{(b-b'')(c-c'')}{bc}(1 - \frac{a}{a})$,
 $\frac{(a-a'')b}{ab}(1 - \frac{c}{c})$, $\frac{(a-a'')c}{ac}(1 - \frac{b}{b})$, $\frac{(b-b'')a}{ab}(1 - \frac{c}{c})$,
 $\frac{(b-b'')c}{bc}(1 - \frac{a}{a})$, $\frac{(a-a'')}{2a}(1 - \frac{b}{b}) \cdot (1 - \frac{c}{c})$, and $\frac{b-b''}{2b} \times$
 $(1 - \frac{a}{a}) \cdot (1 - \frac{c}{c})$:* which being added together and
multiplied by $s(1+p)^{-2}$, will give the expectation of
receiving the sum at the end of the second year. In
like manner we may find the expectation of receiving
the sum at the end of the third and every subsequent
year: and, if these several annual expectations be re-
duced to their lowest terms and arranged under each
other, they will form sixteen collateral series; the sum
of which, continued to the utmost extent of human

* In these, and all similar cases, throughout the remaining part
of this chapter, I suppose it to be an *equal* chance which of the
two lives dies before or after the other during the probable term
of their joint existence. Now, though this is not strictly correct,
(and I have on that account pointed out, in page 139, a method
of finding an approximate value of such chance in order to deter-
mine the value of contingent *Annuities*;) yet in the case of *Assu-
rances* the contingencies are so involved that no material error will
probably arise in any practical cases by thus indiscriminately sup-
posing the chances to be even. The resulting formulæ are hereby
rendered more simple and easy, and are sufficiently accurate for
general use.

life, would be equal to s multiplied into $\frac{1-\rho A}{2(1+\rho)} +$
 $\frac{1-\rho B}{2(1+\rho)} - AB + \frac{1+\dot{A}B}{2(1+\rho)} \cdot \frac{\dot{a}}{a} + \frac{1+\dot{A}B}{2(1+\rho)} \cdot \frac{\dot{b}}{b} + \frac{1+\dot{A}C}{2(1+\rho)} \cdot \frac{\dot{a}}{a} + \frac{1+\dot{B}C}{2(1+\rho)} \cdot \frac{\dot{b}}{b} - \frac{1-\rho ABC}{(1+\rho)} - \frac{ABC}{a} \times$
 $\frac{a}{2a} - \frac{b}{2b} + \frac{c}{2c}$. But this expression, independent of the common multiple s , may be reduced
 to $\frac{2-\rho(A+B-2ABC)+AC+BC}{2(1+\rho)} - AB + \frac{1}{2a} \left[\frac{(\dot{A}B - \dot{A}C)a}{(1+\rho)} - \frac{ABC \cdot a}{a} \right] + \frac{1}{2b} \left[\frac{(\dot{A}B - \dot{B}C)b}{(1+\rho)} - \frac{ABC \cdot b}{b} \right] + \frac{ABC \cdot c}{c}$
 and which, for the sake of a more convenient reference, I shall denote by $\mathbb{D}$.

§ 284. Now, since only part of the several collateral series above mentioned (and which series are represented by the quantities here given) is to be continued to the utmost extent of human life, and as that part will depend on the seniority of the lives concerned; it will be necessary to divide the subsequent investigation of the problem into three distinct cases, according to the seniority of the three lives. For, in each case those several collateral series must be continued only during the existence of the oldest of such lives: because, after that period, we may obtain a more correct value of all the subsequent expectations, by means of the first Lemma in the fifth chapter.

§ 285. *Case 1.* Let A be the oldest of the three lives. In this case, all those collateral series in which the life A is involved must of course be continued to the utmost extent of human life: but all those series in which the life A is not involved must be continued for n terms only. Consequently, the quantities $\frac{1-\rho B}{2(1+\rho)}$, $\frac{1+BC}{2(1+\rho)}$, and $\frac{1+B'C}{2(1+\rho)} \cdot \frac{b}{b}$ will respectively become $\left[\frac{1-\rho B}{2(1+\rho)} - \frac{1-\rho B'}{2(1+\rho)} \cdot \frac{\beta}{b} (1+\rho)^{-n} \right]$, $\left[\frac{1+BC}{2(1+\rho)} - \frac{1+B'C}{2(1+\rho)} \cdot \frac{\beta\gamma}{bc} (1+\rho)^{-n} \right]$ and $\left[\frac{1+B'C}{2(1+\rho)} \cdot \frac{b}{b} - \frac{1+B'C'}{2(1+\rho)} \times \frac{\beta\gamma}{bc} (1+\rho)^{-n} \right]$:* whence, the sum of the first n terms

* See the method of proceeding in these cases explained in Prob. XXII, cor. 2, and Prob. XXVII, cor. 4. I shall, however, here take the opportunity of noticing an error on this subject, into which Mr. Morgan has inadvertently fallen; and which pervades the two papers inserted by him in the 84th and 90th volumes of the *Philosophical Transactions*. In pointing out the method of finding the value of the first n terms of the several series above alluded to, in which the oldest life is not involved, he directs us to convert the quantities, expressing the value of annuities on the *whole* life, into similar expressions for the given *term*. This however is certainly incorrect: for, it is not the *annuities* on those lives which are to be continued for n years, but the *series* from which those annuities arise; and which will in most cases make the value of the annuities, resulting therefrom, equal to a term of $n-1$ years only.

Thus, in the investigation of the present problem in the *Philosophical Transactions* (for 1800, part 1, page 25) he says that, when B is the oldest of the three lives, the annuity on the joint lives AC must be continued for x years: whereas it should be continued

of the several collateral series above mentioned (that is, the sum of all the expectations for the first n years) will become equal to s multiplied into $\mathbb{D} - \frac{1-\rho B}{2(1+\rho)} \times \frac{\beta}{b}(1+\rho)^{-n} - \frac{1+B'C}{2(1+\rho)} \cdot \frac{\beta\gamma}{bc}(1+\rho)^{-n} + \frac{1+B'C}{2(1+\rho)} \times \frac{\beta\gamma}{bc}(1+\rho)^{-n}$.

But, after the decease of A, the expectations, arising from the several contingencies on which the sum depends, may be more correctly expressed for all the subsequent years by means of the first Lemma in Chapter V. For, since the chance of receiving the sum at the end of any one of those years depends on B's dying

for $x-1$ years only; as will readily appear, by attending to the steps of the process. The annuity on the life A, he also says, should be continued for x years: whereas neither x nor $x-1$ will give the correct value in his formula; for, the true value which ought to be there substituted for A is $A' + \frac{1}{r-1} \times \frac{q}{ar}$. As this error runs through the *whole* of the investigations inserted by Mr. Morgan in the two papers above mentioned, I have thought it necessary to advert to it in this place, in order to explain any difference which might be observed between his formulæ and those which are here given.

I cannot dismiss this subject without censuring the careless manner in which Mr. Morgan's papers in the *Philosophical Transactions* have been printed. Were a proof of this charge required, I could not give a more convincing one than the two formulæ in the page above referred to, and which are too erroneous to be of any public utility. I at first imagined that these mistakes might be *errors of the press*; but, when I observe them faithfully copied into the last edition of Dr. Price's treatise, I can be at no loss in attributing them to their proper source.

in the year, and on A having died before C in either of the preceding years (the probability of which latter contingency is, by the Lemma, denoted by ω^*) it follows that the sum of the expectations for those years, continued to the utmost extent of human life, will be equal to $s, \omega \times \left[\frac{\beta - \beta}{b(1+\varphi)^{n+1}} + \frac{\beta - \beta}{b(1+\varphi)^{n+2}} + \frac{\beta - \beta}{b(1+\varphi)^{n+3}} + \&c \right]$: an expression which, by Problem

* Mr. Morgan has given two separate tables for determining the probabilities of survivorship between two lives: the first (which is inserted in *Phil. Trans.* for 1788, page 337) shows the probability of one life dying *before* another; and the second (which is inserted in *Phil. Trans.* for 1794, page 229) shows the probability of one life dying *after* the other. They are both given in the last edition of Dr. Price's *Obs. on Rev. Pay.* vol. 1, page 406, &c.

These tables are deduced from the principles laid down in the first and second Lemma in the fifth Chapter of the present work. The *first* table, here alluded to, will occasionally be found useful in determining the values of ω and φ , which are frequently introduced in the ensuing problems. But as that table is very limited in its extent, I have (in the note in page 114) laid down a method of approximating to those probabilities; which in most probable cases will be found sufficiently correct.

Mr. Morgan might have saved himself the trouble of calculating the *second* table above alluded to; since its application to any practical cases may always be avoided: and moreover, the values in that table are, in my opinion, incorrectly deduced. It is certain that the solutions to the ensuing problems may be more simply expressed by referring *only* to the first table; as any person will readily discover by comparing the formulæ in the present work with those given by Mr. Morgan. See what has been already observed on this subject in the note in page 118.

XXII, cor. 1, is equal to $s.w \times \frac{1-\rho B^o}{(1+\rho)} \cdot \frac{\beta}{b} (1+\rho)^{-n}$.*

Consequently this value, added to the sum of the first n terms of the several collateral series above mentioned, will express the total present value of the given sum in this case required; and which will be found equal to s multiplied into

$$\mathbb{D} - \frac{(\frac{1}{2}-w) \times (1-\rho B^o)\beta + \frac{\gamma}{2c} [(1+B^o C^o)\beta - (1+B^o C^o)\beta']}{b(1+\rho)^{n+1}}.$$

§ 286. *Case 2.* Let B be the oldest of the three lives. In this case all those collateral series in which the life B is involved must be continued to the utmost extent of human life: but all those series in which the life B is not involved must be continued for n terms only. Consequently, the quantities $\frac{1-\rho A}{2(1+\rho)}$, $\frac{1+AC}{2(1+\rho)}$, and $\frac{1+AC}{2(1+\rho)} \cdot \frac{a}{a}$ will respectively become equal to $\left[\frac{1-\rho A}{2(1+\rho)} - \frac{1-\rho A^o}{2(1+\rho)} \cdot \frac{a}{a} (1+\rho)^{-n} \right]$, $\left[\frac{1+AC}{2(1+\rho)} - \frac{1+A^o C^o}{2(1+\rho)} \times \frac{a\gamma}{ac} (1+\rho)^{-n} \right]$ & $\left[\frac{1+AC}{2(1+\rho)} \cdot \frac{a}{a} - \frac{1+A^o C^o}{2(1+\rho)} \cdot \frac{a\gamma}{ac} (1+\rho)^{-n} \right]$: whence, the sum of the first n terms of the several collateral series above mentioned will be equal to s multiplied into $\mathbb{D} - \frac{1-\rho A^o}{2(1+\rho)} \cdot \frac{a}{a} (1+\rho)^{-n} - \frac{1+A^o C^o}{2(1+\rho)} \times \frac{a\gamma}{ac} (1+\rho)^{-n} + \frac{1+A^o C^o}{2(1+\rho)} \cdot \frac{a}{a} (1+\rho)^{-n}$.

But after the decease of B, the expectations, arising

* See the note in page 147.

from the several contingencies on which the sum depends, may be more correctly expressed for all the subsequent years by means of the first Lemma in Chapter V. For, since the chance of receiving the sum at the end of any one of those years depends on A's dying in the year, and on B having died before C in either of the preceding years (the probability of which latter contingency is, by the Lemma, denoted by ϕ) it follows, from what has been said in the last case, that the sum of all the expectations for those years, continued to the utmost extent of human life, will be equal to $s.\phi \times \frac{1-\rho A'}{(1+\rho)} \times \frac{a}{a}(1+\rho)^{-n}$. Consequently this value, added to the sum of the first n terms of the several collateral series above mentioned, will express the total present value of the given sum in this case required; and which will be found equal to s multiplied into

$$D = \frac{(\frac{1}{2}-\phi) \times (1-\rho A')a + \frac{\gamma}{2a} [(1+A'C')a - (1+A'C')a']}{a(1+\rho)^{n+1}}$$

§ 287. *Case 3.* Let C be the oldest of the three lives. In this case, all those collateral series in which the life C is involved must be continued to the utmost extent of human life: but all those series in which the life C is not involved must be continued for n terms only.* Consequently the quantities $\frac{1-\rho A}{2(1+\rho)}$, $\frac{1-\rho B}{2(1+\rho)}$,

* Mr. Morgan, in his investigation of these several cases, has followed his general practice of *changing the symbols* in the last

AB , $\frac{1+AB}{2(1+\rho)} \cdot \frac{a}{a}$, and $\frac{1+AB}{2(1+\rho)} \cdot \frac{b}{b}$ will respectively become equal to $\left[\frac{1-\rho A}{2(1+\rho)} - \frac{1-\rho A^2}{2(1+\rho)} \cdot \frac{a}{a} (1+\rho)^{-n} \right]$, $\left[\frac{1-\rho B}{2(1+\rho)} - \frac{1-\rho B^2}{2(1+\rho)} \cdot \frac{\beta}{b} (1+\rho)^{-n} \right]$, $\left[AB - AB^2 \cdot \frac{\alpha\beta}{ab} (1+\rho)^{-n} \right]$, $\left[\frac{1+AB}{2(1+\rho)} \cdot \frac{a}{a} - \frac{1+AB^2}{2(1+\rho)} \cdot \frac{\alpha\beta}{ab} (1+\rho)^{-n} \right]$ and $\left[\frac{1+AB}{2(1+\rho)} \cdot \frac{b}{b} - \frac{1+AB^2}{2(1+\rho)} \cdot \frac{\alpha\beta}{ab} (1+\rho)^{-n} \right]$: whence, the sum of the first n terms of the several collateral series above mentioned will be equal to s multiplied into $\mathbb{D} - \frac{1-\rho A}{2(1+\rho)} \cdot \frac{a}{a} (1+\rho)^{-n} - \frac{1-\rho B}{2(1+\rho)} \cdot \frac{\beta}{b} (1+\rho)^{-n} + AB \times \frac{\alpha\beta}{ab} (1+\rho)^{-n} - \frac{1+AB^2}{2(1+\rho)} \cdot \frac{\alpha\beta}{ab} (1+\rho)^{-n} - \frac{1+AB^2}{2(1+\rho)} \times \frac{\alpha\beta}{ab} (1+\rho)^{-n}$.

case, and of introducing *another* formula for the solution thereof. But these new formulæ will always be found to be the *same* as those which he has before inserted, *disguised under different characters*. They arise from the *different* modes of summing up the *same* series: agreeably to what I have before remarked in the notes in page 186 and 210.

-In the notation adopted by Mr. Morgan the similarity of the two formulæ is not immediately recognized: and it is possible that he himself was not aware of it. Yet it is singular that he should ever have thought it necessary or useful to recur to *another* expression for the value of the different series, when he had already obtained a formula for that purpose, which (I presume) he must well know was a *general* one. This remark extends to most of the subsequent problems in the present chapter: and will show that, by such acts of supererogation, he has rendered the subject unnecessarily tedious and confused. See the note to the first case of Prob. XLV.

But after the decease of C, the chance of receiving the given sum at the end of any subsequent year will depend on the happening of either of three events: 1° that *either** A or B dies in the year: 2° that only A dies in the year, B having died before C in either of the preceding years: 3° that only B dies in the year, A having died before C in either of the preceding years. The sum of all the expectations of receiving the given sum on the first contingency, for every subsequent year to the utmost extent of human life, is, by Prob. XXII, cor. 1, equal to $s \times \frac{1-\rho A' B'}{(1+\rho)} \times \frac{\alpha \beta}{ab} (1+\rho)^{-n}$;† and the sum of all the expectations on the other two contingencies is, as in the two preceding cases, equal to $s \cdot \phi \times \frac{1-\rho A'}{(1+\rho)} \cdot \frac{\alpha}{a} (1+\rho)^{-n} + s \cdot \omega \times \frac{1-\rho B'}{(1+\rho)} \cdot \frac{\beta}{b} (1+\rho)^{-n}$. Consequently, these values, added to the sum of the first n terms of the several collateral series above mentioned, will express the total present value of the given sum in this case required; and which will be found equal to s multiplied into

$$\begin{aligned} D &= \frac{(\frac{1}{2}-\phi) \times (1-\rho A') \alpha - \frac{\beta}{2b} [(1+A' B') \alpha - (1+A' B') \alpha]}{a(1+\rho)^{n+1}} \\ &- \frac{(\frac{1}{2}-\omega) \times (1-\rho B') \beta - \frac{\alpha}{2a} [(1+A' B') \beta - (1+A' B') \beta]}{b(1+\rho)^{n+1}}. \end{aligned}$$

* Mr. Morgan says (*Phil. Trans.* for 1800, page 26) that this chance depends on *both* the lives becoming extinct within the year; which is evidently erroneous.

† See the note in page 147.

COROLLARY.

§ 288. When the three lives are equal, or all of the same age A , the last three quantities in the formula denoted by $\mathbb{D}$ vanish entirely; and the general expression for this case will become equal to $s \times \frac{1 - \rho(A + AA - AAA)}{(1 + \rho)}$,

SCHOLIUM.

§ 289. As the expressions, deduced from the method of investigation pursued in these three cases, will be frequently referred to in the subsequent problems, it will be convenient to denote them by a more simple character. Let us therefore make

$$w = \frac{(\frac{1}{2} - \pi) \cdot (1 - \rho B')\beta + \frac{\gamma}{2c} [(1 + B'C')\beta - (1 + B'C')\beta']}{b(1 + \rho)^{n+1}}$$

$$x = \frac{(\frac{1}{2} - \phi) \cdot (1 - \rho A')\alpha + \frac{\gamma}{2c} [(1 + A'C')\alpha - (1 + A'C')\alpha']}{a(1 + \rho)^{n+1}}$$

$$y = \frac{(\frac{1}{2} - \pi) \cdot (1 - \rho B')\beta - \frac{\alpha}{2a} [(1 + A'B')\beta - (1 + A'B')\beta']}{b(1 + \rho)^{n+1}}$$

$$z = \frac{(\frac{1}{2} - \phi) \cdot (1 - \rho A')\alpha - \frac{\beta}{2b} [(1 + A'B')\alpha - (1 + A'B')\alpha']}{a(1 + \rho)^{n+1}}$$

Then will the present value of the given sum depending on the contingency mentioned in this problem be respectively expressed by these several formulæ,

$$s[\mathbb{D} - w]$$

$$s[\mathbb{D} - x]$$

$$s[\mathbb{D} - y - z];$$

according as A , B , or C is the oldest of the three lives.

PROBLEM XL.*

§ 290. To determine the present value of a given sum payable on the decease of A or B, provided *either* of them be the *first* or *last* that fails of three given lives A, B, C.

SOLUTION.

The payment of the given sum at the end of the first year will depend on the happening of either of six different events: 1° that all the lives fail in the year: 2° that A and B fail in the year, and that C lives to the end of it: 3° that A and C fail in the year, A having died first, and that B lives to the end of it: 4° that B and C fail in the year, B having died first, and that A lives to the end of it: 5° that only A dies in the year, and that both B and C live to the end of it: 6° that only B dies in the year, and that both A and C live to the end of it. The probabilities of the happening of these several events in the first year are respectively $\frac{(a-a')(b-b')(c-c')}{abc}$, $\frac{(a-a')(b-b')c'}{abc}$, $\frac{(a-a')(c-c')b'}{2abc}$, $\frac{(b-b')(c-c')a'}{2abc}$, $\frac{(a-a')b'c'}{abc}$ and $\frac{(b-b')a'c'}{abc}$; which being added together and multiplied by $s(1+e)^{-1}$, will give the expectation of receiving the sum at the end of the first year.

* Phil. Trans. for 1800, Prob. 3, page 28.

But, in the second and following years, the given sum may be received on the happening of either of nine different events: viz, in addition to the six just enumerated, 7° that A and B die in the year, C having failed in either of the preceding years: 8° that only A fails in the year, and that B and C both fail in either of the preceding years, C having died first: 9° that only B fails in the year, and that A and C both fail in either of the preceding years, C having died first. The probabilities of the happening of these

$$\begin{aligned} & \frac{(a'-a'')(b'-b'')(c'-c'')}{abc}, \quad \frac{(a'-a'')(b'-b'')c''}{abc}, \quad \frac{(a'-a'')(c'-c'')b''}{2abc}, \\ & \frac{(b'-b'')(c'-c'')a''}{2abc}, \quad \frac{(a'-a'')b''c''}{abc}, \quad \frac{(b'-b'')a''c''}{abc}, \quad \frac{(a'-a'')(b'-b'')}{ab} \left(1 - \frac{c'}{c}\right), \\ & \frac{a'-a''}{2a} \left(1 - \frac{b'}{b}\right) \cdot \left(1 - \frac{c'}{c}\right), \quad \text{and} \quad \frac{b'-b''}{2b} \left(1 - \frac{a'}{a}\right) \cdot \left(1 - \frac{c'}{c}\right): \end{aligned}$$

which, being added together and multiplied by $s(1+p)^{-2}$, will give the expectation of receiving the sum at the end of the second year. In like manner, we may find the expectation of receiving the sum at the end of the third and every subsequent year: and, if these several annual expectations be reduced to their lowest terms, and arranged under each other, they will form sixteen collateral series; the sum of which, continued to the utmost extent of human life, would be equal to s multiplied into

$$\begin{aligned} & \frac{1-pA}{2(1+p)} + \frac{1-pB}{2(1+p)} + AB - \frac{1+AB}{2(1+p)} \cdot \frac{a'}{a} \\ & - \frac{1+AB}{2(1+p)} \cdot \frac{b'}{2b} - \frac{1+AC}{2(1+p)} + \frac{1+AC}{2(1+p)} \cdot \frac{a'}{a} - \frac{1+BC}{2(1+p)} + \end{aligned}$$

$$\frac{1+\dot{B}\dot{C}}{2(1+\rho)} \cdot \frac{\dot{b}}{b} + \frac{1-\rho ABC}{(1+\rho)} + \dot{A}BC \cdot \frac{a}{2a} + \dot{A}BC \cdot \frac{\dot{b}}{2b} -$$

$$\dot{A}BC \cdot \frac{\dot{c}}{c} : \text{an expression which, independent of the}$$

common multiple s , may be reduced to the formula

$$\frac{2-\rho(A+B+2ABC)-AC-BC}{2(1+\rho)} + AB - \frac{1}{2a} \left[\frac{(\dot{A}\dot{B}-\dot{A}\dot{C})\dot{a}}{(1+\rho)} \right.$$

$$\left. - \dot{A}BC \cdot \frac{a}{b} \right] - \frac{1}{2b} \left[\frac{(\dot{A}\dot{B}-\dot{B}\dot{C})\dot{b}}{(1+\rho)} - \dot{A}BC \cdot \frac{b}{c} \right] - \dot{A}BC \cdot \frac{\dot{c}}{2c};$$

and which, for the sake of a more convenient reference I shall denote by $\mathbb{E}$.

§ 291. *Case 1.* Let A be the oldest of the three lives. In this case, the sum of the first n terms of the several collateral series above mentioned will, by pursuing the same steps as in the first case of the preceding problem, be equal to $\mathbb{E} - \frac{1-\rho B^*}{2(1+\rho)} \cdot \frac{\beta}{b} (1+\rho)^{-n} +$

$$\frac{1+B^*C^*}{2(1+\rho)} \cdot \frac{\beta\gamma}{bc} (1+\rho)^{-n} - \frac{1+B^*C^*}{2(1+\rho)} \cdot \frac{\beta\gamma}{bc} (1+\rho)^{-n}.$$

After the decease of A, however, the expectations arising from these contingencies may be more correctly expressed for all the subsequent years by means of the second Lemma in the fifth chapter: for, since the chance of receiving the sum at the end of any one of those years depends on B's dying in the year and on A having died after C in either of the preceding years (the probability of which latter contingency is, by the Lemma, denoted by $1-\varpi$), we shall find that the sum of the expectations for those years, continued to

the utmost extent of human life, will be equal to $s.(1-\varpi) \times \frac{1-\varrho B'}{(1+\varrho)} \cdot \frac{\beta}{b}(1+\varrho)^{-n}$. Consequently this value, added to the sum of the first n terms of the several collateral series above mentioned, will make the total present value of the given sum to be equal to $s[\mathbb{E} + w]$.

§ 292. *Case 2.* Let B be the oldest of the three lives. In this case, the sum of the first n terms of the several collateral series above mentioned will, by pursuing the same steps as in the second case to the preceding problem, be now equal to $\mathbb{E} - \frac{1-\varrho A'}{2(1+\varrho)} \times \frac{a}{a}(1+\varrho)^{-n} + \frac{1+A'C'}{2(1+\varrho)} \cdot \frac{a\gamma}{ac}(1+\varrho)^{-n} - \frac{1+A'C'}{2(1+\varrho)} \times \frac{a\gamma}{ac}(1+\varrho)^{-n}$. After the decease of B, however, the expectation arising from these contingencies may be more correctly expressed for all the subsequent years by means of the Lemma above mentioned: for, since the chance of receiving the sum at the end of any one of those years depends on A's dying in the year, and on B having died after C in either of the preceding years, (the probability of which is, by the Lemma, denoted by $1-\phi$) we shall find that the sum of the expectations for those years, continued to the utmost extent of human life, will be equal to $s.(1-\phi) \times \frac{1-\varrho A'}{(1+\varrho)} \cdot \frac{a}{a}(1+\varrho)^{-n}$. Consequently, this value added

to the sum of the first n terms as above found, will express the total present value of the given sum in this case required. Whence, such present value will be equal to $s[\mathbb{E} + x]$.

§ 293. *Case 3.* Let C be the oldest of the three lives. In this case, the sum of the first n terms of the several collateral series above mentioned, will (agreeably to the method pursued in the third case to the preceding problem) be equal to $\mathbb{E} - \frac{1-\rho A'}{2(1+\rho)} \times \frac{\alpha}{a}(1+\rho)^{-n} - \frac{1-\rho B'}{2(1+\rho)} \cdot \frac{\beta}{b}(1+\rho)^{-n} - AB' \cdot \frac{\alpha\beta}{ab}(1+\rho)^{-n} + \frac{1+A'B'}{2(1+\rho)} \cdot \frac{\alpha\beta}{ab}(1+\rho)^{-n} + \frac{1+A'B'}{2(1+\rho)} \cdot \frac{\alpha\beta}{ab}(1+\rho)^{-n}$. But, after the decease of C, the expectations arising from these contingencies may be more correctly expressed for all the subsequent years by means of the Lemma above mentioned: for, the chance of receiving the sum at the end of any one of those years will depend on the happening of either of three events: 1° that A and B both fail in the year: 2° that A fails in the year, B having died after C in either of the preceding years: 3° that B fails in the year, A having died after C in either of the preceding years. The probabilities of the happening of these three events in the $(n+1)^{\text{st}}$ year, are respectively $\frac{(\alpha-a')(\beta-\beta')}{ab}$, $\frac{\alpha-a'}{a}(1-\phi-\frac{\beta}{b})^*$ and $\frac{\beta-\beta'}{b}(1-\omega-\frac{\alpha}{a})^*$ which being added together

* See the Scholium in page 121.

and multiplied by $(1+\rho)^{-(n+1)}$ will give the expectation of receiving the sum at the end of the $(n+1)^{\text{st}}$ year.

In like manner, the probabilities of the happening of these events in the $(n+2)^{\text{nd}}$ year are respectively $\frac{(\alpha'-\alpha'').(\beta'-\beta'')}{ab}$, $\frac{\alpha'-\alpha''}{a}(1-\phi-\frac{\beta'}{b})$ and $\frac{\beta'-\beta''}{b}(1-\omega-\frac{\alpha'}{a})$; which being added together and multiplied by $(1+\rho)^{-(n+2)}$ will give the expectation of receiving the sum at the end of the $(n+2)^{\text{nd}}$ year. By the same method of reasoning we may find the expectation of receiving the sum at the end of the $(n+3)^{\text{rd}}$ year: and so on to the utmost extent of human life; the sum of all which annual expectations, when reduced to their lowest terms, will be found equal to s multiplied into $(1-\phi) \times \frac{1-\rho A'}{(1+\rho)} \cdot \frac{\alpha}{a} (1+\rho)^{-n} + (1-\omega) \times \frac{1-\rho B'}{(1+\rho)} \cdot \frac{\beta}{b} (1+\rho)^{-n} - \frac{1+A'B'}{(1+\rho)} \cdot \frac{\alpha\beta}{ab} (1+\rho)^{-n} + A'B' \cdot \frac{\alpha\beta}{ab} (1+\rho)^{-n}$: which, being added to the sum of the first n terms of the several collateral series above mentioned, will express the total present value of the given sum in this case required. Whence, such present value will be equal to $s[\text{E} + y + z]$.

COROLLARY.

§ 294. When all the lives are equal, or of

the same age A , the last three quantities in the formula denoted by $\mathbb{E}$ vanish entirely; and the general expression for this case will become $s \times \frac{1 - \rho(A - AA + AAA)}{(1 + \rho)}$.

SCHOLIUM.

§ 295. Since it appears that the present value of the given sum, payable on the contingency mentioned in this problem, added to the present value of the same sum, payable on the contingency mentioned in the preceding problem, are together equal to $s \times \frac{2 - \rho(A + B)}{(1 + \rho)}$: it follows that, the value of one of them being determined, the other may be easily obtained by subtracting such value from the general expression here given: provided the ages of the lives are the same in both instances.

PROBLEM XLI.*

§ 296. To determine the present value of a given sum payable on the decease of A , provided he be the *second* that dies of three given lives A , B , C , and provided C dies before B .†

* Simpson's *Sup. Prob.* 38, and Prob. 4, in page 74. Dodson, vol. iii. Ques. 42 to 47. Morgan, Prob. 18; and in Phil. Trans. for 1789, page 41.

† That is, provided C dies first, and A second of the three lives; or, provided C be then dead and B living: but I have thought it best, for the sake of uniformity, to preserve the mode of expression adopted in the text.

SOLUTION.

The chance of receiving the given sum at the end of the first year will depend on the happening of either of two events: 1° that all the lives fail in that year, C having died first and A next: 2° that A and C both fail in that year A having died last, and that B lives to the end of it. The probabilities of the happening of these two events are respectively $\frac{(a-a')(b-b')(c-c')}{6abc}$,

and $\frac{(a-a')(c-c')b}{2abc}$; which, being added together and multiplied by $s(1+p)^{-1}$, will give the expectation of receiving the sum at the end of the first year.

But the chance of receiving the given sum at the end of the second and following years will depend on the happening of either of four different events: 1° that all the lives fail in the year, C having died first and A next: 2° that A and C both fail in the year, A having died last, and that B lives to the end of it: 3° that both A and B fail in the year, A having died first, and that C dies in either of the preceding years: 4° that only A dies in the year, B living to the end of it, and C having died in either of the preceding years. The probabilities of the happening of these several events in the second year are respectively $\frac{(a-a'')(b-b'')(c-c'')}{6abc}$, $\frac{(a-a'')(c-c'')b}{2abc}$, $\frac{(a-a'')(b-b'')}{2ab}(1-\frac{c'}{c})$, and $\frac{(a-a'')b''}{ab}(1-\frac{c'}{c})$: which being added together and

multiplied by $s(1+p)^{-2}$, will give the expectation of receiving the sum at the end of the second year. In like manner we may find the expectation of receiving the sum at the end of the third and every subsequent year, to the utmost extent of human life; the sum of all which yearly values will be the total present value of the given sum to be received on the above contingency.

These several yearly expectations being reduced to their least terms, and arranged under each other, will form twelve collateral series: the sum of all which, or the present value required, will be found equal to $s[A^B - A^{BC}]$.*

* This problem is one of frequent occurrence; and is the first of that series of problems, inserted by Mr. Morgan in the *Philosophical Transactions*, for determining the value of contingent reversions where three lives are involved in the survivorship. Mr. Morgan has, in his usual manner, given *two* formulæ for the solution of the problem; leading us thereby to imagine that the value of the given sum depends on the *seniority* of the lives concerned. This however is not the fact: for, the theorem which I have given in the text is a *general* one; and will apply to either case.

As it may be satisfactory to the reader to see the different results which are obtained from the present investigation and from that pursued by Mr. Morgan, I shall here subjoin his formulæ. But since the same lives are not involved in the same contingencies, I shall (in order to prevent any misunderstanding on this subject) prefix his own statement of the problem.

“ To determine the value of a given sum payable on the contingency of C's surviving B, provided the life of A shall be then extinct.

“ When either B or C is the oldest of the three lives, the

COROLLARY.

§ 297. When the lives are all equal, or of the same age A , this expression (agreeably to what has been said in the corollary to Prob. XXX) will become equal to $s \times \frac{1-\rho(3AA-2AAA)}{6(1+\rho)}$: that is, equal to one

value of the given sum will be $= S$ into $\frac{x}{6c} \times \left[\frac{\beta(FK-AFK)}{b} - (BK-ABK) \right] + \frac{\beta}{3b} \times (FC-AFC) - \frac{r-1}{3r} \times (BC-ABC) - \frac{m}{3br} \times (PC-APC) + \frac{d}{6cr} \times \left[(BT-ABT) - \frac{m(PT-APT)}{b} \right]$.

"When A is the bldest of the three lives, the value will be $= S$ into $\Sigma - \frac{x}{3c} \times \left[\frac{\alpha(HK-HBK)}{a} + \frac{AK-ABK}{2} \right] - \frac{\alpha}{6a} \times (HC-HBC) - \frac{r-1}{3r} \times (AC-ABC) + \frac{s}{6ar} \times (NC-NBC) + \frac{d}{3cr} \times \left[\frac{AT-ABT}{2} + \frac{s(NT-NBT)}{a} \right]$:" where Σ denotes the same as B^c in the notation used in the present treatise, and the other characters the same as already explained in the notes in page 215 and 221.

These formulæ are taken from the last edition of Dr. Price's *Obs. on Rev. Pay.* vol. 1, page 392: the gross and careless errors therein being first corrected. It is hardly necessary to remark (after the repeated observations that I have made of a similar kind) that all those expressions in the second formula, and some of those in the first, which involve *two* joint lives are totally useless: since they might be made to vanish, in the manner pointed out in page 216. They consequently now only tend to make the arithmetical solution unnecessarily tedious and confused: and the reader perhaps may be surprised to learn that these two formulæ are precisely the same, disguised under different symbols!!!

sixth of the present value of the given sum payable on the extinction of any two out of the three given lives.

PROBLEM XLII.*

§ 298. To determine the present value of a given sum payable on the decease of A; provided he be the *last* that dies of three given lives A, B, C, and provided C dies before B.†

SOLUTION.

The given sum cannot be received at the end of the first year unless all the three lives fail in that year, and in the order above mentioned; the probability of which is $\frac{(a-a')(b-b')(c-c')}{6abc}$; which being multiplied by $s(1+e)^{-1}$, will give the expectation of receiving the sum at the end of the first year.

But in the second and following years, the given sum may be received on the happening of either of three different events: 1° that all the lives fail in the year, in the order above mentioned: 2° that A and B both fail in the year, A having died last, and C having failed in either of the preceding years: 3° that only

* Morgan, Prob. 29; and in Phil. Trans. for 1794, Prob. 6, page 253.

† That is, provided C dies first, B next, and A last.

A dies in the year, and that both B and C fail in either of the preceding years, C having died first.

The probabilities of the happening of these several events

in the second year, are respectively $\frac{(a'-a'')(b'-b'')(c'-c'')}{6abc}$,

$\frac{(a'-a'')(b'-b'')}{2ab}(1-\frac{c'}{c})$, and $\frac{a'-a''}{2a}(1-\frac{b'}{b})\cdot(1-\frac{c'}{c})$:

which, being added together and multiplied by $s(1+\rho)^2$, will give the expectation of receiving the sum at the end of the second year. In like manner we may find the expectation of receiving the sum at the end of the third and every subsequent year during the continuance of the oldest of the given lives: and, if these several annual expectations be reduced to their lowest terms and arranged under each other, they will form fourteen collateral series; the first n terms of which will vary according to the seniority of the lives concerned.

§ 299. *Case 1.* Let A be the oldest of the three lives. In this case the several series here alluded to must be continued to the utmost extent of human life; because the life A is involved in each collateral series. Therefore the value of such series will be equal to s multiplied into $\frac{1-\rho A}{2(1+\rho)} + \frac{AB}{2} - AB \cdot \frac{a}{2a} - \frac{1+AC}{2(1+\rho)} + \frac{1+AC}{2(1+\rho)} \cdot \frac{a}{a} + \frac{1-\rho ABC}{6(1+\rho)} - \frac{1+ABC}{6(1+\rho)} \cdot \frac{a}{a} + ABC \cdot \frac{a}{6a} + \frac{1+ABC}{3(1+\rho)} \cdot \frac{b}{b} + ABC \cdot \frac{b}{6b} - \frac{1+ABC}{6(1+\rho)} \cdot \frac{c}{c} - ABC \cdot \frac{c}{3c}$: an expression which, independent of the common multiple

s , may be reduced to $\frac{1-\rho(3A+ABC)-3AC}{6(1+\rho)} + \frac{AB}{2} + \frac{1}{6a} \left[\frac{(2+3AC-ABC)a}{(1+\rho)} - (3AB - ABC)a \right] + \frac{1}{6b} \times \left[\frac{(1+ABC)2b}{(1+\rho)} + ABC.b \right] - \frac{1}{6c} \left[\frac{(1+ABC)c}{(1+\rho)} + ABC.2c \right]$; and which I shall denote by $\mathbb{F}$. Therefore when A is the oldest of the three lives, the present value of the given sum will be represented by $s \times \mathbb{F}$.

§ 300. *Case 2.* Let B be the oldest of the three lives. In this case the sum of the first n terms of the several collateral series above mentioned will be found equal to $\mathbb{F} - \frac{1-\rho A}{2(1+\rho)} \cdot \frac{a}{a} (1+\rho)^{-n} + \frac{1+AC}{2(1+\rho)} \times \frac{a\gamma}{ac} (1+\rho)^{-n} - \frac{1+AC}{2(1+\rho)} \cdot \frac{a\gamma}{ac} (1+\rho)^{-n}$. But, after the extinction of the life of B, the sum of the expectations for all the subsequent years may be more correctly expressed as in the second case of Prob. XL, by $s.(1-\phi) \times \frac{1-\rho A}{(1+\rho)} \cdot \frac{a}{a} (1+\rho)^{-n}$. Therefore if this value be added to the sum of the first n terms of the several series just found, it will make the total present value of the given sum, when B is the oldest of the three lives, equal to $s[\mathbb{F} + x]$.

§ 301. *Case 3.* Let C be the oldest of the three

A dies in the year, and that both B and C fail in either of the preceding years, C having died first.

The probabilities of the happening of these several events

in the second year, are respectively $\frac{(a-a'')(b-b'')(c-d)}{6abc}$,

$\frac{(a-a'')(b-b'')}{2ab} \left(1 - \frac{c}{c}\right)$, and $\frac{a-a''}{2a} \left(1 - \frac{b}{b}\right) \cdot \left(1 - \frac{c}{c}\right)$:

which, being added together and multiplied by $s(1+\rho)^2$, will give the expectation of receiving the sum at the end of the second year. In like manner we may find the expectation of receiving the sum at the end of the third and every subsequent year during the continuance of the oldest of the given lives: and, if these several annual expectations be reduced to their lowest terms and arranged under each other, they will form fourteen collateral series; the first n terms of which will vary according to the seniority of the lives concerned.

§ 299. *Case 1.* Let A be the oldest of the three lives. In this case the several series here alluded to must be continued to the utmost extent of human life; because the life A is involved in each collateral series. Therefore the value of such series will be equal to s multiplied into $\frac{1-\rho A}{2(1+\rho)} + \frac{AB}{2} - AB \cdot \frac{a}{2a} - \frac{1+AC}{2(1+\rho)} + \frac{1+AC}{2(1+\rho)} \cdot \frac{a}{a} + \frac{1-\rho ABC}{6(1+\rho)} - \frac{1+ABC}{6(1+\rho)} \cdot \frac{a}{a} + ABC \cdot \frac{a}{6a} + \frac{1+ABC}{3(1+\rho)} \cdot \frac{b}{b} + ABC \cdot \frac{b}{6b} - \frac{1+ABC}{6(1+\rho)} \cdot \frac{c}{c} - ABC \cdot \frac{c}{3c}$: an expression which, independent of the common multiple

s , may be reduced to $\frac{1-\rho(3A+ABC)-3AC}{6(1+\rho)} + \frac{AB}{2} + \frac{1}{6a} \left[\frac{(2+3AC-ABC)a}{(1+\rho)} - (3AB-ABC)a \right] + \frac{1}{6b} \times \left[\frac{(1+ABC)2b}{(1+\rho)} + ABC.b \right] - \frac{1}{6c} \left[\frac{(1+ABC)c}{(1+\rho)} + ABC.2c \right]$; and which I shall denote by $\mathbb{F}$. Therefore when A is the oldest of the three lives, the present value of the given sum will be represented by $s \times \mathbb{F}$.

§ 300. *Case 2.* Let B be the oldest of the three lives. In this case the sum of the first n terms of the several collateral series above mentioned will be found equal to $\mathbb{F} - \frac{1-\rho A}{2(1+\rho)} \cdot \frac{a}{a} (1+\rho)^{-n} + \frac{1+AC}{2(1+\rho)} \times \frac{ay}{ac} (1+\rho)^{-n} - \frac{1+AC}{2(1+\rho)} \cdot \frac{ay}{ac} (1+\rho)^{-n}$. But, after the extinction of the life of B, the sum of the expectations for all the subsequent years may be more correctly expressed as in the second case of Prob. XL, by $s(1-\phi) \times \frac{1-\rho A}{(1+\rho)} \cdot \frac{a}{a} (1+\rho)^{-n}$. Therefore if this value be added to the sum of the first n terms of the several series just found, it will make the total present value of the given sum, when B is the oldest of the three lives, equal to $s[\mathbb{F} + x]$.

§ 301. *Case 3.* Let C be the oldest of the three

lives. In this case the sum of the first n terms of the several collateral series above alluded to will be found equal to $\mathbb{F} - \frac{1-\rho A'}{2(1+\rho)} \cdot \frac{\alpha}{a} (1+\rho)^{-n} - AB' \cdot \frac{\alpha\beta}{2ab} (1+\rho)^{-n}$

+ $AB' \cdot \frac{\alpha\beta}{2ab} (1+\rho)^{-n}$. But, after the extinction of the life of C, the chance of receiving the given sum at the end of any one year will depend on the happening of two events only: 1° that A and B both fail in the year, A having died last: 2° that only A fails in the year, B having died after C in either of the preceding years: the probabilities of the happening of these events in the $(n+1)^{\text{st}}$ year are respectively $\frac{(\alpha-\alpha')(\beta-\beta')}{2ab}$ and $\frac{\alpha-\alpha'}{a} (1-\phi - \frac{\beta}{b})$, which being mul-

tiplied by $(1+\rho)^{-n+1}$ will give the expectation of receiving the sum at the end of the $(n+1)^{\text{st}}$ year. In like manner we might proceed to find the expectations of receiving the given sum for very subsequent year to the utmost extent of human life: and the sum of all those expectations will be found equal to s multi-

plied into $(1-\phi) \times \frac{1-\rho A'}{(1+\rho)} \cdot \frac{\alpha}{a} (1+\rho)^{-n} - \frac{1+AB'}{2(1+\rho)} \times \frac{\alpha\beta}{ab} (1+\rho)^{-n} + AB' \cdot \frac{\alpha\beta}{2ab} (1+\rho)^{-n} + \frac{1+AB'}{2(1+\rho)} \cdot \frac{\alpha\beta}{ab} \times (1+\rho)^{-n} - AB' \cdot \frac{\alpha\beta}{2ab} (1+\rho)^{-n}$. Consequently this

value, added to the sum of the first n terms of the several collateral series above mentioned, will express the total present value of the given sum, in this case

required. Whence, such present value will be equal to $s[\mathbb{F} + z]$.*

COROLLARY.

§ 302. When the three lives are equal, or all of the same age A , the last three quantities in the formulæ denoted by $\mathbb{F}$ vanish altogether, and the general expression in this case will become $s \times \frac{1 - \rho(3A - 3AA + AAA)}{6(1 + \rho)}$: that is, equal to one sixth of the present value of the given sum, payable on the extinction of the longest of the three lives.

PROBLEM XLIII.†

§ 303. To determine the present value of a given sum payable on the decease of A , provided he be the *first* or *second* that fails of three given lives A, B, C ; and provided C , in the latter case, dies before B .

* Mr. Morgan has divided this problem (I know not for what reason) into *four* distinct cases: and in his investigations of these cases he has fallen into the same errors and absurdities that I have so frequently noticed in the preceding pages. But as it is not my wish to swell the present work with those disgusting repetitions; and as the remarks, which I have already made, apply with nearly equal propriety to *all* Mr. Morgan's problems, I shall not here enter more into detail. It may however be useful perhaps for Mr. Morgan to know that it is *impossible* that L (or the value of an annuity on the longest of the three lives) can *properly* arise in *any* of his problems, unless those lives are *equal*.

† Phil. Trans. for 1800, Prob. 4, page 32.

SOLUTION.

The payment of the sum at the end of the first year will depend on the happening of either of four different events: 1° that all the lives fail in the year, A having died first, or C having died first and A second: 2° that A and B fail in the year, A having died first, and that C lives to the end of it: 3° that A and C die in the year, and that B lives to the end of it: 4° that only A dies in the year, and that B and C both live to the end of it. The probabilities of the happening of these several events are respectively $\frac{(a-a')(b-b')(c-c')}{2abc}$, $\frac{(a-a')(b-b')c'}{2abc}$, $\frac{(a-a')(c-c')b'}{abc}$, and $\frac{(a-a')b'c'}{abc}$: which, being added together and multiplied by $s(1+p)^{-1}$, will give the expectation of receiving the sum at the end of the first year.

But in the second and following years, the given sum may be received on the happening of either of six different events; viz. in addition to the four already enumerated, 5° that A and B fail in the year, A having died first, and that C dies in either of the preceding years: 6° that only A dies in the year, B living to the end of it, and C having died in either of the preceding years. The probabilities of the happening of these six different events in the second year are respectively $\frac{(a-a'')(b-b'')(c-c'')}{2abc}$, $\frac{(a-a'')(b-b'')c''}{2abc}$, $\frac{(a-a'')(c-c'')b''}{abc}$, $\frac{(a-a'')b''c''}{abc}$, $\frac{(a-a'')(b-b'')}{2ab}(1-\frac{c'}{c})$, and

$\frac{(a-a'')b}{ab} \left(1 - \frac{c}{c'}\right)$: which being added together, and multiplied by $s(1+p)^{-t}$, will give the expectation of receiving the sum at the end of the second year. And so for every subsequent year to the utmost extent of human life: the sum of all which values will be the total present value of the sum required: and which will be found equal to $s \times A^B$, or to the present value of the given sum payable on the decease of A provided he be the first that dies of the two lives A, B; as found by Prob. XXVII. The truth of which is evident: for, the payment of the given sum can be prevented only by the event of B dying before A.*

PROBLEM XLIV.†

§ 304. To determine the present value of a given sum payable on the decease of A, provided he be the *second* or *third* that fails of three given lives A, B, C; and provided C dies before B,

SOLUTION.

The given sum may be received at the end of the first year, on the happening of either of two events: 1^o that all the lives become extinct in that year, C

* It may be here useful to remark that the present value found by this problem is equal to the sum of the two values found by Prob. XXIX and XLI: and their agreement in this particular confirms the accuracy of the investigation.

† Phil. Trans. for 1800, Prob. 5, page 33.

having died first: 2° that A and C both fail in that year, C having died first, and that B lives to the end of it. The probabilities of the happening of these events are respectively $\frac{(a-a')(b-b')(c-c')}{3abc}$, and $\frac{(a-a')(c-c')b}{2abc}$; which, being added together and multiplied by $s(1+p)^{-1}$, will give the expectation of receiving the sum at the end of the first year.

But, in the second and following years the given sum may be received on the happening of either of five different events: 1° that all the lives fail in the year, C having died first: 2° that A and C both fail in the year, C having died first, and that B lives to the end of it: 3° that A and B both die in the year, C having died in either of the preceding years: 4° that only A dies in the year, B living to the end of it, and C having died in either of the preceding years; 5° that only A fails in the year, and that B and C both fail in either of the preceding years, C having died first. The probabilities of the happening of these several events in the second year are respectively $\frac{(a-a'')(b-b'')(c-c'')}{3abc}$, $\frac{(a-a'')(c-c'')b}{2abc}$, $\frac{(a-a'')(b-b'')}{ab}(1-\frac{c'}{c})$, $\frac{(a-a'')b}{ab}(1-\frac{c'}{c})$, and $\frac{(a-a'')}{2a}(1-\frac{b'}{b})\cdot(1-\frac{c'}{c})$: which, being added together and multiplied by $s(1+p)^{-2}$, will give the expectation of receiving the sum at the end of the second year. In like manner we may find the expectation of receiving the sum at the end of the third and every succeeding year: and if these

several annual expectations be reduced to their lowest terms and arranged under each other, they will form fourteen collateral series; the sum of the first n terms of which will vary according to the seniority of the lives concerned,

§ 305. *Case 1.* Let A be the oldest of the three lives. In this case, the terms of the several series here alluded to must be continued to the utmost extent of human life, because the life of A is involved in each collateral series: therefore the sum of it will be equal to s multiplied into $\frac{1-\rho A}{2(1+\rho)} + \frac{1+AB}{2(1+\rho)} - \frac{1+AB}{2(1+\rho)} \cdot \frac{a}{a} - \frac{1+AC}{2(1+\rho)} + \frac{1+AC}{2(1+\rho)} \cdot \frac{a}{a} - \frac{1-\rho ABC}{6(1+\rho)} + \frac{1+ABC}{6(1+\rho)} \cdot \frac{a}{a} - ABC \cdot \frac{a}{6a} + \frac{1+BC}{6(1+\rho)} \cdot \frac{b}{b} + ABC \cdot \frac{b}{3b} - \frac{1+ABC}{3(1+\rho)} \cdot \frac{c}{c} - ABC \cdot \frac{c}{6c}$: an expression which, independent of the common multiple s , may be reduced, as in the former cases, to $\frac{2-\rho(3A-ABC)+3AB-3AC}{6(1+\rho)} + \frac{1}{6a} \times \left[\frac{(1-3AB+3AC+ABC)a}{(1+\rho)} - ABC \cdot a \right] + \frac{1}{6b} \left[\frac{(1+ABC)b}{(1+\rho)} + ABC \cdot 2b \right] - \frac{1}{6c} \left[\frac{(1+ABC)2c}{(1+\rho)} + ABC \cdot c \right]$; and which I shall denote by G . Therefore, when A is the oldest life, the required value will be represented by $s \times G$.

§ 306. *Case 2.* Let B be the oldest of the three lives. In this case the first n terms of the several collateral series above mentioned will be equal to $G - \frac{1-\rho A'}{2(1+\rho)} \cdot \frac{\alpha}{a} (1+\rho)^{-n} + \frac{1+A'C'}{2(1+\rho)} \cdot \frac{\alpha\gamma}{ac} (1+\rho)^{-n} - \frac{1+A'C'}{2(1+\rho)} \cdot \frac{\alpha\gamma}{ac} (1+\rho)^{-n}$. But after the decease of B the expectations for all the subsequent years may be more correctly expressed by means of the first Lemma in the fifth Chapter: for, since the chance of receiving the sum at the end of any one of those years depends on A's failing in the year, C having died before B in either of the preceding years (the probability of which is, by the Lemma denoted by $1-\phi$) we shall find that the sum of the expectations for those years continued to the utmost extent of human life, will be equal to $s \cdot (1-\phi) \times \frac{1-A'}{(1+\rho)} \cdot \frac{\alpha}{a} (1+\rho)^{-n}$. Consequently, this value added to the sum of the first n terms of the series above found, will express the total present value of the given sum in this case required; and which will be found equal to $s[G + x]$.

§ 307. *Case 3.* Let C be the oldest of the three lives. In this case the first n terms of the several collateral series above mentioned will be equal to $G - \frac{1-\rho A'}{2(1+\rho)} \cdot \frac{\alpha}{a} (1+\rho)^{-n} - \frac{1+A'B'}{2(1+\rho)} \cdot \frac{\alpha\beta}{ab} (1+\rho)^{-n} + \frac{1+A'B'}{2(1+\rho)} \cdot \frac{\alpha\beta}{ab} (1+\rho)^{-n}$. But, after the decease of C,

the chance of receiving the given sum at the end of any subsequent year will depend solely on the death of A, C having died before B in either of the preceding years; whence, the sum of the expectations for all the subsequent years will, as in the preceding case, be denoted by $s(1-\phi) \times \frac{1-A^n}{(1+\rho)} \cdot \frac{a}{a} (1+\rho)^{-n}$. Consequently, this value, added to the sum of the first n terms of the series above found, will represent the total present value of the given sum in this case required: and which will be found equal to $s[G+z]$.*

COROLLARY.

§ 308. When the three lives are equal, or all of the same age A, the last three quantities in the formula denoted by G destroy each other, and the general expression will in this case become $s \times \frac{2-\rho(3A-AAA)}{6(1+\rho)}$.

PROBLEM XLV.†

§ 309. To determine the present value of a given sum payable on the decease of A, provided he be the *first* or *last* that fails of three given lives A, B, C; and provided C, in the latter case, dies before B.

* It may be here useful to remark that the present value found by this problem is equal to the sum of the two values found by Prob. XLI and XLII: and their agreement in this particular is a proof of the accuracy of the investigation.

† Phil. Trans. for 1800, Prob. 6, page 37.

SOLUTION.

The given sum may be received at the end of the first year on the happening of either of four different events: 1° that all the lives fail in that year, A having died first, or C having died first, B second, and A last: 2° that A and B fail in that year, A having died first, and that C lives to the end of it: 3° that A and C fail in that year, A having died first, and that B lives to the end of it: 4° that only A dies in that year, and that B and C both live to the end of it. The probabilities of the happening of these several events are respectively $\frac{(a-a')(b-b')(c-c')}{2abc}$, $\frac{(a-a')(b-b')c'}{2abc}$, $\frac{(a-a')(c-c')b'}{2abc}$, and $\frac{(a-a')b'c'}{abc}$: which, being added together and multiplied by $s(1+p)^{-1}$, will give the expectation of receiving the sum at the end of the first year.

But, in the second and following years the given sum may be received on the happening of either of six different events: 1° that all the lives fail in the year, A having died first, or C having died first, B second, and A last: 2° that A and B fail in the year, A having died first, and that C lives to the end of it: 3° that A and C fail in the year, A having died first, and that B lives to the end of it: 4° that only A dies in the year, and that B and C both live: 5° that A and B both fail in the year, A having died last, and C having failed in either of the preceding

years: 6° that only A fails in the year, and that B and C both fail in either of the preceding years, C having died first. The probabilities of the happening of these several events in the second year are respectively

$$\frac{(a'-a'')(b'-b'')(c'-c'')}{2abc}, \quad \frac{(a'-a'')(b'-b'')c''}{2abc}, \quad \frac{(a'-a'')(c'-c'')b''}{2abc},$$

$$\frac{(a'-a'')b''c''}{abc}, \quad \frac{(a'-a'')(b'-b'')}{2ab}\left(1-\frac{c'}{c}\right), \quad \text{and} \quad \frac{a'-a''}{2a}\left(1-\frac{b'}{b}\right) \times$$

$$\left(1-\frac{c'}{c}\right): \text{ which, being added together and multiplied}$$

by $s(1+\rho)^{-2}$, will give the expectation of receiving the sum at the end of the second year. In like manner, we may find the expectation of receiving the sum at the end of the third and every subsequent year, during the continuance of the oldest of the given lives: and, if these several annual expectations be reduced to their lowest terms, and arranged under each other, they will form twelve collateral series; the first n terms of which will vary according to the seniority of the lives concerned.

§ 310. Case 1. Let A be the oldest of the three lives. In this case, the several collateral series here mentioned must be continued to the utmost extent of human life, because the life A is involved in every term of it: therefore the sum of it will be equal to s multiplied into $\frac{1-\rho^A}{2(1+\rho)} + \frac{AB}{2} - AB \cdot \frac{a}{2a} - \frac{1+AC}{2(1+\rho)} +$
 $\frac{1+AC}{2(1+\rho)} \cdot \frac{a}{a} + \frac{1-\rho ABC}{2(1+\rho)} - \frac{1+ABC}{2(1+\rho)} \cdot \frac{a}{a} + ABC \cdot \frac{a}{2a} +$

$\frac{1+ABC}{2(1+\rho)} \cdot \frac{b}{b} - ABC \cdot \frac{c}{2c}$: an expression which, independent of the common multiple s , may be reduced to $\frac{1-\rho(A+ABC)-AC}{2(1+\rho)} + \frac{AB}{2} + \frac{1}{2a} \left[\frac{(AC-ABC)a}{(1+\rho)} - (AB-ABC)a \right] + \frac{1+ABC}{2(1+\rho)} \cdot \frac{b}{b} - ABC \cdot \frac{c}{2c}$; and which I shall denote by **II**. Therefore, when A is the oldest of the three lives, the value required will be equal to $s \times \text{II}$.*

* It is really amusing to observe the manner in which Mr. Morgan has summed up the several series arising from this problem: and since this manner is by no means unusual with him, and as it gives me an opportunity of explaining more fully what I have already said, in the preceding pages, on his singular and confused method of investigating such cases, I shall here (for the last time) enter once more on the subject.

He makes the total value of the series, continued to the utmost extent of human life, equal to S into $\frac{r-1}{2r} (V-A-ABC) + \frac{AC}{2} - \frac{AB}{2r} + \frac{md}{2bcr} (1+APT) - \frac{d}{2cr} (1+AT+BT-ABT) - \frac{\beta\kappa}{2bc} \times AFK + \frac{\beta}{2b} (AF+FC-AFC)$: See *Phil. Trans.* for 1800, page 37. Now, this is certainly the correct value of the series: but, it is necessary to remark that $\frac{\beta \cdot FC}{2b}$ is equal to $\frac{d}{2cr} (1+BT)$; and, as these two quantities are used with *contrary signs*, it is evident that they ought not to have been introduced into the equation. Consequently the formula should have been reduced to $\frac{r-1}{2r} \times (V-A-ABC) + \frac{AC}{2} - \frac{AB}{2r} + \frac{md}{2bcr} (1+APT) - \frac{d}{2cr} \times (AT-ABT) - \frac{\beta\kappa}{2bc} \times AFK + \frac{\beta}{2b} (AF-AFC)$.

The reader will observe that this is the total value of the series,

§ 311. *Case 2.* Let B be the oldest of the three lives. In this case, the sum of the first n terms of the several collateral series above mentioned will be equal to $H - \frac{1-\rho A}{2(1+\rho)} \cdot \frac{a}{a} (1+\rho)^{-n} + \frac{1+A'C}{2(1+\rho)} \times \frac{a\gamma}{ac} (1+\rho)^{-n} - \frac{1+A'C}{2(1+\rho)} \cdot \frac{a\gamma}{ac} (1+\rho)^{-n}$. But, after the decease of B, the value of the expectations for all the sub-

continued to the utmost extent of human life; and therefore that it will truly denote the value of the given sum *when A is the oldest of the three lives*. Mr. Morgan, however, when he comes to consider that case, says that the symbols *must be changed*; and he then makes the given sum equal to S into $\frac{r-1}{2r}(V-A-ABC) + \frac{AC}{2} - \frac{AB}{2r} - \frac{x.AK}{2c} + \frac{a.HBC}{2a} - \frac{d}{2cr}(AT-ABT) + \frac{s}{2ar}(1+NC+NB-NBC) - \frac{ds}{2acr}(1+NBT)$. Now, a moment's attention to this formula will show us that it does not in reality differ from the preceding one: for

$$\begin{aligned} \frac{md}{2bcr}(1+APT) &= \frac{a.HBC}{2a} \\ \frac{\beta}{2b} \left(AF - \frac{x}{c} AFK \right) &= \frac{s}{2ar}(NB-NBC) \\ \frac{\beta.AFC}{2b} &= \frac{ds}{2acr}(1+NBT); \end{aligned}$$

whence, these values may be safely substituted for each other, since they by no means denote *different* quantities, but arise merely from the different methods of summing up the *same* series, agreeably to what I have before remarked. But, as the first formula had been already deduced by him, and was stated at length, it certainly did not become necessary to confuse the subject by introducing a second.

The most remarkable circumstance, however, attending this

sequent years may be more correctly expressed, as in the second case to Prob. XL, by $s(1-\varphi)\frac{1-\varphi A^0}{(1+\varphi)}\cdot\frac{a}{a}(1+\varphi)^{-n}$: and which being added to the sum of the first n terms found as above, will make the total present value of the given sum, when B is the oldest of the three lives, equal to $s[\text{H} + x]$.

investigation, is that (after discarding the two useless quantities $\frac{\beta.FC}{2b} - \frac{d}{2cr}(1+BT)$ from this second formula) he should insert in their stead two *other* useless quantities, $\frac{x.AK}{2c}$ and $\frac{s}{2ar}(1+NC)$: which, being equal to each other, and used with *contrary signs*, ought of course to have been discarded also. Surely Mr. Morgan could not have been writing either for the *convenience* or the *information* of mankind!!!

I shall here take the opportunity of repeating the observation made in the note in page 219; that, although the quantities here alluded to are strictly and mathematically *equal*, and being used with contrary signs *destroy each other*, yet (owing to the imperfection of the Tables which show the value of Life Annuities) they oftentimes produce a *real* expression when solved *arithmetically*. Thus $\frac{s}{ar}(1+NC) - \frac{x.AK}{c}$ (which, expressed by the characters made use of in this work, becomes $\frac{1+AC}{(1+\varphi)}\cdot\frac{a}{a} - AC\cdot\frac{1}{c}$) is, from what has been said in the note in page 186, equal to 0: but if reduced to *numbers* (taking the age of A 15 years, the age of C 18 years, the rate of interest 4 per cent, and the probabilities of living as at *Northampton*) it will become equal to $13.362 - 13.348 = .014$. This fact will sufficiently show how necessary it is to discard all *superfluous* quantities: and that Mr. Morgan's formulæ are, at best, but approximations.

§ 312. *Case 3.* Let C be the oldest of the three lives. In this case, the first n terms of the several collateral series above mentioned will be equal to $\text{HI} - \frac{1-\rho A'}{2(1+\rho)} \cdot \frac{\alpha}{a} (1+\rho)^{-n} - AB' \cdot \frac{\alpha\beta}{2ab} (1+\rho)^{-n} + AB' \cdot \frac{\alpha\beta}{2ab} (1+\rho)^{-n}$. But, after the decease of C the chance of receiving the given sum at the end of any subsequent year will depend on the happening of either of two events: 1° that A dies after B in the year: 2° that only A dies in the year, B having died after C in either of the preceding years. The sum of the expectations on the happening of these events has been already found, by the third case in Prob. XLII, to be equal to $(1-\phi) \times \frac{1-\rho A'}{(1+\rho)} \cdot \frac{\alpha}{a} (1+\rho)^{-n} - \frac{1+A'B'}{2(1+\rho)} \times \frac{\alpha\beta}{ab} (1+\rho)^{-n} + AB' \cdot \frac{\alpha\beta}{2ab} (1+\rho)^{-n} + \frac{1+A'B'}{2(1+\rho)} \cdot \frac{\alpha\beta}{ab} \times (1+\rho)^{-n} - AB' \cdot \frac{\alpha\beta}{2ab} (1+\rho)^{-n}$: which, being added to the sum of the first n terms of the several collateral series above mentioned, will make the total present value required equal to $s[\text{HI} + z]$.*

COROLLARY.

§ 313. When the three lives are equal, or all of

* It may be useful to remark that the present value found by this problem is equal to the sum of the two values found by Prob. XXIX and XLII; which confirms the accuracy of the investigation.

the same age A , the last three quantities in the formula denoted by H destroy each other, and the general expression in this case will become equal to

$$s \times \frac{1 - \rho(A - AA + AAA)}{2(1 + \rho)}.$$

PROBLEM XLVI.*

§ 314. To determine the present value of a given sum payable on the decease of A and B , provided they be the *first* that fail of three given lives A, B, C .

SOLUTION.

In order to receive the sum at the end of the first year, either of these two events must happen: 1° that all the lives fail in that year, C having died last: 2° that the two lives A and B fail in that year, and that C lives to the end of it. The probabilities of the happening of these two events are respectively $\frac{(a-a')(b-b')(c-c')}{3abc}$, and $\frac{(a-a')(b-b')c'}{abc}$: which, being added together and multiplied by $s(1+\rho)^{-1}$, will give the expectation of receiving the sum at the end of the first year.

But, in the second and following years, the given sum may be received on the happening of either of six different events: 1° that all the lives fail in the year,

* Simpson's *Sup. Prob.* 39. Dodson, vol. iii. Ques. 48. Morgan, *Prob.* 22; and in *Phil. Trans.* for 1791, *Prob.* 4, page 258.

C having died last: 2° that A and B both fail in the year, and that C lives to the end of it: 3° that A and C both fail in the year, A having died first, and that B dies in either of the preceding years: 4° that B and C both fail in the year, B having died first, and that A dies in either of the preceding years: 5° that only A dies in the year, C living to the end of it, and B having died in either of the preceding years: 6° that only B dies in the year, C living to the end of it, and A having died in either of the preceding years. The probabilities of the happening of these several events in the second year are respectively $\frac{(a'-a'')(b'-b'')(c'-c'')}{3abc}$, $\frac{(a'-a'')(b'-b'')c''}{abc}$, $\frac{(a'-a'')(c'-c'')}{2ac}(1-\frac{b'}{b})$, $\frac{(b'-b'')(c'-c'')}{2bc} \times (1-\frac{a'}{a})$, $\frac{(a'-a'')c''}{ac}(1-\frac{b'}{b})$, and $\frac{(b'-b'')c''}{bc}(1-\frac{a'}{a})$: which being added together and multiplied by $s(1+p)^{-2}$, will give the expectation of receiving the sum at the end of the second year. In like manner we may find the expectation of receiving the sum at the end of the third and every subsequent year, to the utmost extent of human life: and if these several expectations be reduced to their lowest terms and arranged under each other they will form sixteen collateral series, the sum of which will be found equal to $s[\mathcal{A}^c + B^c - \mathcal{B}^c]$.

COROLLARY.

§ 315. When the lives are all equal to each other,

or of the same age A , this expression (agreeably to what has been said in the corollary to Prob. XXX) will become equal to $s \times \frac{1-\rho(3AA-2AAA)}{3(1+\rho)}$: that is, equal to one third of the present value of the given sum payable on the extinction of any two out of the three given lives.

PROBLEM XLVII.*

§ 316. To determine the present value of a given sum payable on the decease of A and B , provided they be the *last* that fail of three given lives A, B, C .

SOLUTION.

It is evident that the given sum cannot be received at the end of the first year unless all the lives become extinct, C having died first: the probability of which is $\frac{(a-a')(b-b')(c-c')}{3abc}$: and which, being multiplied by $s(1+\rho)^{-1}$, will give the expectation of receiving the sum at the end of the first year.

But, in the second and following years, the given sum may be received on the happening of either of four different events: 1^o that all the lives fail in the year, C having died first: 2^o that A and B fail in the year, C having died in either of the preceding years:

* Morgan, Prob. 23; and in Phil. Trans. for 1794, Prob. 5, page 247.

3° that only A fails in the year, and that B and C both fail in either of the preceding years, B having died last: 4° that only B fails in the year, and that A and C both fail in either of the preceding years, A having died last. The probabilities of the happen-

ing of these several events, in the second year, are respectively $\frac{(a-a'')(b-b'')(c-c'')}{3abc}$, $\frac{(a-a'')(b-b'')}{ab}(1-\frac{c'}{c})$,

$\frac{a-a''}{2a}(1-\frac{b'}{b})\cdot(1-\frac{c'}{c})$, and $\frac{b-b''}{2b}(1-\frac{a'}{a})\cdot(1-\frac{c'}{c})$:

which, being added together and multiplied by $s(1+\rho)^{-2}$, will give the expectation of receiving the sum at the end of the second year. In like manner, we may find the expectation of receiving the sum at the end of the third and every subsequent year: and, if these several expectations be reduced to their lowest terms and arranged under each other, they will form nineteen collateral series; the sum of which, if continued to the utmost extent of human life, would be equal to s multiplied into

$$\begin{aligned} & \frac{1-\rho A}{2(1+\rho)} + \frac{1-\rho B}{2(1+\rho)} + AB - \frac{1+AB}{2(1+\rho)} \cdot \frac{a'}{a} - \\ & \frac{1+AB}{2(1+\rho)} \cdot \frac{b'}{b} - \frac{1+AC}{2(1+\rho)} + \frac{1+AC}{2(1+\rho)} \cdot \frac{a'}{a} - \frac{1+BC}{2(1+\rho)} + \frac{1+BC}{2(1+\rho)} \times \\ & \frac{b'}{b} + \frac{1-\rho ABC}{3(1+\rho)} + \frac{1+ABC}{6(1+\rho)} \cdot \frac{a'}{a} + ABC \cdot \frac{a'}{3a} + \frac{1+ABC}{6(1+\rho)} \times \\ & \frac{b'}{b} + ABC \cdot \frac{b'}{3b} - \frac{1+ABC}{3(1+\rho)} \cdot \frac{c'}{c} - ABC \cdot \frac{2c'}{3c}: \text{ an ex-} \\ & \text{pression which, independent of the common multiple} \\ & s, \text{ may be reduced to } \frac{2-\rho(3A+3B+2ABC)-3AC-3BC}{6(1+\rho)} \end{aligned}$$

$$\begin{aligned}
 &+ AB + \frac{1}{6a} \left[\frac{(1-3AB+3AC+ABC)a}{(1+\rho)} + ABC.2a \right] \\
 &+ \frac{1}{6b} \left[\frac{(1-3AB+3BC+ABC)b}{(1+\rho)} + ABC.2b \right] - \frac{1}{3c} \times \\
 &\left[\frac{(1+ABC)c}{(1+\rho)} + ABC.2c \right]; \text{ and which I shall denote} \\
 &\text{by II.}
 \end{aligned}$$

§ 317. *Case 1.* Let A be the oldest of the three lives. In this case, the sum of the first n terms of the several collateral series above mentioned will be equal to $\text{II} - \frac{1-\rho B'}{2(1+\rho)} \cdot \frac{\beta}{b} (1+\rho)^{-n} + \frac{1+B'C'}{2(1+\rho)} \times \frac{\beta\gamma}{bc} (1+\rho)^{-n} - \frac{1+B'C'}{2(1+\rho)} \cdot \frac{\beta\gamma}{bc} (1+\rho)^{-n}$. But, after the decease of A, the sum of the expectations for all the subsequent years will, as in the first case of Prob. XL, be more correctly expressed by $s(1-\varpi) \times \frac{1-\rho B'}{(1+\rho)} \times \frac{\beta}{b} (1+\rho)^{-n}$: and which, being added to the first n terms just found, will make the total present value, when A is the oldest life, equal to $s[\text{II} + w]$.

§ 318. *Case 2.* Let B be the oldest of the three lives. In this case, the sum of the first n terms of the several collateral series above mentioned will be equal to $\text{II} - \frac{1-\rho A'}{2(1+\rho)} \cdot \frac{a}{a} (1+\rho)^{-n} + \frac{1+A'C'}{2(1+\rho)} \times$

$\frac{a\gamma}{ac}(1+\rho)^{-n} - \frac{1+A'C}{2(1+\rho)} \cdot \frac{a\gamma}{ac}(1+\rho)^{-n}$. But, after the decease of B, the sum of the expectations for all the subsequent years will, as in the second case of Prob. XL, be more correctly expressed by $s(1-\phi) \times \frac{1-\rho A}{(1+\rho)} \cdot \frac{a}{a}(1+\rho)^{-n}$: and which, being added to the first n terms just found, will make the total present value, when B is the oldest life, equal to $s[\text{II} + x]$.

§ 319. *Case 3.* Let C be the oldest of the three lives. In this case, the sum of the first n terms of the several collateral series above mentioned will be equal to $\text{II} - \frac{1-\rho A}{2(1+\rho)} \cdot \frac{a}{a}(1+\rho)^{-n} - \frac{1-\rho B}{2(1+\rho)} \cdot \frac{\beta}{b}(1+\rho)^{-n} - AB \cdot \frac{a\beta}{ab}(1+\rho)^{-n} + \frac{1+A'B}{2(1+\rho)} \cdot \frac{a\beta}{ab}(1+\rho)^{-n} + \frac{1+A'B}{2(1+\rho)} \times \frac{a\beta}{ab}(1+\rho)^{-n}$. But, after the decease of C, the sum of the expectations for all the subsequent years will, as in the third case of Prob. XL, be more correctly expressed by s multiplied into $(1-\phi) \times \frac{1-\rho A}{(1+\rho)} \times \frac{a}{a}(1+\rho)^{-n} + (1-\omega) \times \frac{1-\rho B}{(1+\rho)} \cdot \frac{\beta}{b}(1+\rho)^{-n} - \frac{1+A'B}{(1+\rho)} \times \frac{a\beta}{ab}(1+\rho)^{-n} + AB \cdot \frac{a\beta}{ab}(1+\rho)^{-n}$: and which, being added to the first n terms just found, will make the total present value, when C is the oldest life, equal to $s[\text{II} + y + z]$.

COROLLARY.

§ 320. When all the lives are equal, or of the same age A , the last three quantities in the formula denoted by II destroy each other, and the general expression in this case will become equal to $s \times \frac{1-\rho(3A-3AA+AAA)}{3(1+\rho)}$: that is, equal to one third of the present value of the given sum to be received on the extinction of the longest of the three lives.

PROBLEM XLVIII.*

§ 321. To determine the present value of a given sum payable on the decease of A and B , provided they be the *first* and *last* that fail of three given lives A , B , C .

SOLUTION.

The given sum cannot be received at the end of the first year, unless all the three lives are extinct, C having died second; the probability of which is $\frac{(a-a')(b-b')(c-c')}{3abc}$: and which being multiplied by $s(1+\rho)^{-1}$, will give the expectation of receiving the sum at the end of the first year.

* This case appears to have been omitted by Mr. Morgan: and no solution of it, prior to the present one, has hitherto been published.

But, in the second and following years, the given sum may be received on the happening of either of five different events: 1° that all the lives fail in the year, C having died second: 2° that A and C both fail in the year, A having died last, and B having failed in either of the preceding years: 3° that B and C both fail in the year, B having died last, and A having failed in either of the preceding years: 4° that only A dies in the year, and that B and C both fail in either of the preceding years, B having died first: 5° that only B dies in the year, and that A and C both fail in either of the preceding years, A having died first. The probabilities of the happening of these several events in the second year are respectively $\frac{(a-a'')(b-b'')(c-c'')}{3abc}$, $\frac{(a-a')(c-c')}{2ac}(1-\frac{b'}{b})$, $\frac{(b-b')(c-c')}{2bc} \times (1-\frac{a'}{a})$, $\frac{a-a''}{2a}(1-\frac{b'}{b}) \cdot (1-\frac{c'}{c})$, and $\frac{b-b''}{2b}(1-\frac{a'}{a}) \times (1-\frac{c'}{c})$: which being added together, and multiplied by $s(1+\rho)^{-2}$, will give the expectation of receiving the sum at the end of the second year. In like manner we may find the expectation of receiving the sum at the end of the third and every subsequent year: and if these several annual expectations be reduced to their lowest terms, and arranged under each other, they will form nineteen collateral series; the sum of which, if continued to the utmost extent of human life, would be equal to s multiplied into $\frac{1-\rho^A}{2(1+\rho)} + \frac{1-\rho^B}{2(1+\rho)} -$

$$\begin{aligned} & \frac{1+AB}{(1+\rho)} + \frac{1+AB}{2(1+\rho)} \cdot \frac{a}{a} + \frac{1+AB}{2(1+\rho)} \cdot \frac{b}{b} + \frac{AC}{2} - AC \cdot \frac{a}{2a} + \\ & \frac{BC}{2} - BC \cdot \frac{b}{2b} + \frac{1-\rho ABC}{3(1+\rho)} - \frac{1+ABC}{3(1+\rho)} \cdot \frac{a}{a} - ABC \cdot \frac{a}{6a} \\ & - \frac{1+ABC}{3(1+\rho)} \cdot \frac{b}{b} - ABC \cdot \frac{b}{6b} + \frac{1+ABC}{3(1+\rho)} \cdot \frac{2c}{c} + ABC \cdot \frac{c}{3c}. \end{aligned}$$

But this expression, independent of the common multiplier s , may be reduced to $\frac{2-\rho(3A+3B+2ABC)-6AB}{6(1+\rho)} +$

$$\begin{aligned} & \frac{AC+BC}{2} + \frac{1}{6a} \left[\frac{(1+3AB-2ABC)a}{(1+\rho)} - (3AC+ABC)a \right] \\ & + \frac{1}{6b} \left[\frac{(1+3AB-2ABC)b}{(1+\rho)} - (3BC+ABC)b \right] + \frac{1}{3c} \times \\ & \left[\frac{(1+ABC)2c}{(1+\rho)} + ABC.c \right]; \text{ and I shall denote it by } K. \end{aligned}$$

§ 322. *Case 1.* Let A be the oldest of the three lives. In this case, the first n terms of the several collateral series, here alluded to, will be equal to $K - \frac{1-\rho B}{2(1+\rho)} \cdot \frac{\beta}{b} (1+\rho)^{-n} - BC \cdot \frac{\beta\gamma}{2bc} (1+\rho)^{-n} + BC \cdot \frac{\beta\gamma}{2bc} (1+\rho)^{-n}$. But, after the decease of A, the chance of receiving the sum at the end of any subsequent year will depend on the happening of two different events: 1° that B and C both fail in the year, B having died last: 2° that only B dies in the year, C having died after A in either of the preceding years. The probabilities of the happening of these two events in the $(n+1)^{\text{st}}$ year, are respectively $\frac{(\beta-\beta')(\gamma-\gamma')}{2bc}$, and

$\frac{\beta - \beta'}{b}(\varpi - \frac{\gamma}{c})$; * which being added together and multiplied by $s(1 + \rho)^{-(n+1)}$ will give the expectation of receiving the sum at the end of the $(n+1)^{\text{st}}$ year.

In like manner, the probabilities of the happening of these two events in the $(n+2)^{\text{nd}}$ year, are respectively $\frac{(\beta - \beta')(\gamma' - \gamma'')}{2bc}$, and $\frac{\beta - \beta''}{b}(\varpi - \frac{\gamma'}{c})$; which being added together and multiplied by $s(1 + \rho)^{-(n+2)}$ will give the expectation of receiving the sum at the end of the $(n+2)^{\text{nd}}$ year: and so on for all the subsequent years to the utmost extremity of human life. But, the sum of all these annual expectations will be found to be equal to s multiplied into $\varpi \times \frac{1 - \rho^{B'}}{(1 + \rho)} \times \frac{\beta}{b}(1 + \rho)^{-n} - \frac{1 + B'C'}{2(1 + \rho)} \cdot \frac{\beta\gamma}{bc}(1 + \rho)^{-n} + B'C' \times \frac{\beta\gamma}{2bc}(1 + \rho)^{-n} + \frac{1 + B'C''}{2(1 + \rho)} \cdot \frac{\beta\gamma}{bc}(1 + \rho)^{-n} - B'C'' \times \frac{\beta\gamma}{2bc}(1 + \rho)^{-n}$: and which, being added to the first n terms of the several collateral series above mentioned, will make the total present value of the given sum, when A is the oldest life, equal to $s[K - w]$.

§ 323. Case 2. Let B be the oldest of the three lives. In this case, the first n terms of the several collateral series above mentioned will be equal to

* See the Scholium in page 122.

$$\begin{aligned} & \frac{1+AB}{(1+\rho)} + \frac{1+AB}{2(1+\rho)} \cdot \frac{a}{a} + \frac{1+AB}{2(1+\rho)} \cdot \frac{b}{b} + \frac{AC}{2} - AC \cdot \frac{a}{2a} + \\ & \frac{BC}{2} - BC \cdot \frac{b}{2b} + \frac{1-\rho ABC}{3(1+\rho)} - \frac{1+ABC}{3(1+\rho)} \cdot \frac{a}{a} - ABC \cdot \frac{a}{3a} \\ & - \frac{1+ABC}{3(1+\rho)} \cdot \frac{b}{b} - ABC \cdot \frac{b}{3b} + \frac{1+ABC}{3(1+\rho)} \cdot \frac{2c}{c} + ABC \cdot \frac{c}{3c}. \end{aligned}$$

But this expression, independent of the common multiplier s , may be reduced to $\frac{2-\rho(3A+3B+2ABC)-6AB}{6(1+\rho)} +$

$$\begin{aligned} & \frac{AC+BC}{2} + \frac{1}{6a} \left[\frac{(1+3AB-2ABC)a}{(1+\rho)} - (3AC+ABC)a \right] \\ & + \frac{1}{6b} \left[\frac{(1+3AB-2ABC)b}{(1+\rho)} - (3BC+ABC)b \right] + \frac{1}{3c} \times \\ & \left[\frac{(1+ABC)2c}{(1+\rho)} + ABC.c \right]; \text{ and I shall denote it by } K. \end{aligned}$$

§ 322. *Case 1.* Let A be the oldest of the three lives. In this case, the first n terms of the several collateral series, here alluded to, will be equal to $K - \frac{1-\rho B}{2(1+\rho)} \cdot \frac{\beta}{b} (1+\rho)^{-n} - BC \cdot \frac{\beta\gamma}{2bc} (1+\rho)^{-n} + BC \cdot \frac{\beta\gamma}{2bc} (1+\rho)^{-n}$. But, after the decease of A, the chance of receiving the sum at the end of any subsequent year will depend on the happening of two different events: 1° that B and C both fail in the year, B having died last: 2° that only B dies in the year, C having died after A in either of the preceding years. The probabilities of the happening of these two events in the $(n+1)^{\text{st}}$ year, are respectively $\frac{(\beta-\beta').(\gamma-\gamma')}{2bc}$, and

$\frac{\beta - \beta'}{b}(\varpi - \frac{\gamma}{e})$;* which being added together and multiplied by $s(1 + \rho)^{-(n+1)}$ will give the expectation of receiving the sum at the end of the $(n+1)^{\text{st}}$ year.

In like manner, the probabilities of the happening of these two events in the $(n+2)^{\text{nd}}$ year, are respectively $\frac{(\beta - \beta')(\gamma' - \gamma'')}{2bc}$, and $\frac{\beta - \beta''}{b}(\varpi - \frac{\gamma'}{e})$; which being added together and multiplied by $s(1 + \rho)^{-(n+2)}$ will give the expectation of receiving the sum at the end of the $(n+2)^{\text{nd}}$ year: and so on for all the subsequent years to the utmost extremity of human life. But, the sum of all these annual expectations will be found to be equal to s multiplied into $\varpi \times \frac{1 - \rho^B}{(1 + \rho)} \times \frac{\beta}{b}(1 + \rho)^{-n} - \frac{1 + B'C}{2(1 + \rho)} \cdot \frac{\beta\gamma}{bc}(1 + \rho)^{-n} + B'C \times \frac{\beta\gamma}{2bc}(1 + \rho)^{-n} + \frac{1 + B'C}{2(1 + \rho)} \cdot \frac{\beta\gamma}{bc}(1 + \rho)^{-n} - B'C \times \frac{\beta\gamma}{2bc}(1 + \rho)^{-n}$: and which, being added to the first n terms of the several collateral series above mentioned, will make the total present value of the given sum, when A is the oldest life, equal to $s[K - w]$.

§ 323. *Case 2.* Let B be the oldest of the three lives. In this case, the first n terms of the several collateral series above mentioned will be equal to

* See the Scholium in page 122.

$$K - \frac{1-\rho A'}{2(1+\rho)} \cdot \frac{a}{a} (1+\rho)^{-n} - AC' \cdot \frac{a\gamma}{2ac} (1+\rho)^{-n} +$$

$$AC' \cdot \frac{a\gamma}{2ac} (1+\rho)^{-n}.$$
 But, after the decease of B, the chance of receiving the sum at the end of any subsequent year will depend on the happening of two different events: 1° that A and C both fail in the year, A having died last: 2° that only A dies in the year, C having died after B in either of the preceding years. The probabilities of the happening of these two events in the $(n+1)^{\text{st}}$ year, are respectively $\frac{(a-a')(\gamma-\gamma')}{2ac}$, and $\frac{a-a'}{a}(\phi - \frac{\gamma}{c})$; which being added together and multiplied by $s(1+\rho)^{-(n+1)}$ will give the expectation of receiving the sum at the end of the $(n+1)^{\text{st}}$ year.

In like manner, we might find the expectation of receiving the sum at the end of the $(n+2)^{\text{nd}}$ and every subsequent year, to the utmost extremity of human life; the sum of all which annual expectations will be found equal to s multiplied into $\phi \times \frac{1-\rho A'}{(1+\rho)} \times \frac{a}{a} (1+\rho)^{-n} - \frac{1+AC'}{2(1+\rho)} \cdot \frac{a\gamma}{ac} (1+\rho)^{-n} + AC' \times \frac{a\gamma}{2ac} (1+\rho)^{-n} + \frac{1+AC'}{2(1+\rho)} \cdot \frac{a\gamma}{ac} (1+\rho)^{-n} - AC' \times \frac{a\gamma}{2ac} (1+\rho)^{-n}$: and which, being added to the first n terms of the several collateral series above mentioned, will make the total present value of the given sum, when B is the oldest of the three lives, equal to $s[K - x]$.

§ 324. *Case 3.* Let C be the oldest of the three lives. In this case, the sum of the first n terms of the several collateral series above mentioned will be equal to

$$\begin{aligned} K - \frac{1-\rho^A}{2(1+\rho)} \cdot \frac{a}{a} (1+\rho)^{-n} - \frac{1-\rho^B}{2(1+\rho)} \cdot \frac{\beta}{b} (1+\rho)^{-n} + \\ \frac{1+A'B'}{(1+\rho)} \cdot \frac{a\beta}{ab} (1+\rho)^{-n} - \frac{1+A'B'}{2(1+\rho)} \cdot \frac{a\beta}{ab} (1+\rho)^{-n} - \\ \frac{1+A'B'}{2(1+\rho)} \cdot \frac{a\beta}{ab} (1+\rho)^{-n}. \end{aligned}$$

But, after the decease of C, the chance of receiving the sum at the end of any subsequent year, will depend on the happening of two different events: 1° that only A dies in the year, C having died after B in either of the preceding years (the probability of which latter contingency will now be denoted by ϕ): 2° that only B dies in the year, C having died after A in either of the preceding years (the probability of which latter contingency will now be denoted by ϖ). Consequently, the sum of the expectations on the happening of these events, continued to the utmost extent of human life, will be equal to $s \cdot \phi \times \frac{1-\rho^A}{(1+\rho)} \cdot \frac{a}{a} (1+\rho)^{-n} + s \cdot \varpi \times \frac{1-\rho^B}{(1+\rho)} \times \frac{\beta}{b} (1+\rho)^{-n}$: and which, being added to the sum of the first n terms of the several collateral series above mentioned, will make the total present value of the given sum, when C is the oldest life, equal to $s[K - y - z]$.*

* It may not be improper here to remark that the present values of the sums, found by Prob. XLVI, XLVII, and XLVIII, are together equal to $s \times \frac{1-\rho(A+B-AB)}{(1+\rho)}$; or to the present value of the given sum after the longest of the two lives A, B.

COROLLARY.

§ 325. When the lives are equal, or all of the same age A , the last three quantities in the formula denoted by $\mathbf{K}$ destroy each other, and the general expression in this case will become $s \times \frac{1-\rho(3A-3AA+AAA)}{3(1+\rho)}$: that is, equal to one third of the present value of the given sum to be received on the extinction of the longest of the three lives.

PROBLEM XLIX.*

§ 326. To determine the present value of a given sum payable on the decease of A and B , provided B dies before another life C .

SOLUTION.

The payment of the sum at the end of the first year will depend on the happening of either of two events: 1° that all the lives fail in the year, B having died before C : 2° that only A and B die in that year, C living to the end of it. The probabilities of the happening of these two events are respectively $\frac{(a-a')(b-b')(c-c')}{2abc}$ and $\frac{(a-a')(b-b')c'}{abc}$: which being added together, and multiplied by $s(1+\rho)^{-1}$, will give the expectation of receiving the sum at the end of the first year.

* Morgan, Prob. 28; and in Phil. Trans. for 1794, Prob. 3, page 242.

But, in the second and following years the given sum may be received on the happening of either of seven different events: 1° that all the lives fail in the year, B having died before C: 2° that A and B fail in the year, and that C lives to the end of it: 3° that A and C both fail in the year, and that B dies in either of the preceding years: 4° that B and C both fail in the year, B having died first, and that A dies in either of the preceding years: 5° that only A dies in the year, B having failed in either of the preceding years, and C living to the end of it: 6° that only B dies in the year, C living to the end of it, and A having died in either of the preceding years: 7° that only A dies in the year, and that B and C both fail in either of the preceding years, B having died first. The probabilities of the happening of these several events in the second year are respectively $\frac{(a-a'')(b-b'')(c-c'')}{2abc}$, $\frac{(a-a'')(b-b'')c}{abc}$, $\frac{(a-a'')(c-c'')}{ac}(1-\frac{b}{b})$, $\frac{(b-b'')(c-c'')}{2bc} \times$ $(1-\frac{a}{a})$, $\frac{(a-a'')c}{ac}(1-\frac{b}{b})$, $\frac{(b-b'')c}{bc}(1-\frac{a}{a})$, and $\frac{a-a''}{2a}(1-\frac{b}{b}) \cdot (1-\frac{c}{c})$: which, being added together and multiplied by $s(1+p)^{-2}$, will give the expectation of receiving the sum at the end of the second year. In like manner may be found the expectation of receiving the sum at the end of the third and every subsequent year, during the continuance of the oldest life: and if these yearly expectations be reduced to their lowest

terms and arranged under each other, they will form fourteen collateral series; the first n terms of which will vary according to the seniority of the lives concerned.

§ 327. *Case 1.* Let A be the oldest of the three lives. In this case the first n terms of the several collateral series above mentioned will be equal to the sum of the terms continued to the utmost extent of human life; because the life of A is involved in each collateral series: therefore its value will be equal to s multiplied into $\frac{1-\rho A}{2(1+\rho)} - \frac{1+AB}{2(1+\rho)} + \frac{1+AB}{2(1+\rho)} \cdot \frac{a'}{a} + \frac{1+AC}{2(1+\rho)} - \frac{1+AC}{2(1+\rho)} \cdot \frac{a'}{a} + \frac{1-\rho BC}{2(1+\rho)} - \frac{1+BC}{2(1+\rho)} \cdot \frac{b'}{b} + BC \cdot \frac{b'}{2b} - \frac{1-\rho ABC}{2(1+\rho)} - ABC \cdot \frac{b'}{2b} + ABC \cdot \frac{c'}{2c}$. But this expression may be reduced to $\frac{1-\rho(A+BC-ABC)-AB+AC}{2(1+\rho)} + \frac{(AB-AC)a'}{2(1+\rho)a} - \frac{1}{2b} \left[\frac{(1+BC)b'}{(1+\rho)} - (BC-ABC) \cdot \frac{b'}{b} \right] + ABC \cdot \frac{c'}{2c}$: and which I shall denote by $\mathbb{L}$; consequently, when A is the oldest life, the present value required will be represented by $s \times \mathbb{L}$.

§ 328. *Case 2.* Let B be the oldest of the three lives. In this case the first n terms of the several collateral series above mentioned will be equal to $\mathbb{L} - \frac{1-\rho A}{2(1+\rho)} \cdot \frac{a}{a} (1+\rho)^{-n} - \frac{1+AC}{2(1+\rho)} \cdot \frac{a\gamma}{ac} (1+\rho)^{-n} +$

$\frac{1+A^C}{2(1+\rho)} \cdot \frac{\alpha\gamma}{ac} (1+\rho)^{-n}$. But after the decease of B the sum of the expectations for all the subsequent years, continued to the utmost extent of human life, will (as in the second case of Prob. XXXIX) be equal to $s \cdot \phi \times \frac{1-\rho A}{(1+\rho)} \cdot \frac{\alpha}{a} (1+\rho)^{-n}$: which being added to the sum of the first n terms of the several collateral series above mentioned, will make the total present value of the given sum equal to $s[\mathbf{L} - x]$.

§ 329. *Case 3.* Let C be the oldest of the three lives. In this case the first n terms of the several collateral series above alluded to will be equal to $\mathbf{L} - \frac{1-\rho A}{2(1+\rho)} \cdot \frac{\alpha}{a} (1+\rho)^{-n} + \frac{1+A^B}{2(1+\rho)} \cdot \frac{\alpha\beta}{ab} (1+\rho)^{-n} - \frac{1+A^B}{2(1+\rho)} \cdot \frac{\alpha\beta}{ab} (1+\rho)^{-n}$. But, after the decease of C, the sum of the expectations for all the subsequent years may be more correctly expressed as in the preceding case, by $s \cdot \phi \times \frac{1-\rho A}{(1+\rho)} \cdot \frac{\alpha}{a} (1+\rho)^{-n}$: and which being added to the first n terms of the several collateral series above mentioned will make the total present value of the given sum, in this case required, equal to $s[\mathbf{L} - z]$.

COROLLARY.

§ 330. When the three lives are equal to each other, or of the same age A, the last three quantities

in the formula denoted by $\mathbb{L}$ destroy each other, and the general expression will in this case become equal to $s \times \frac{1-\rho(A+AA-AAA)}{2(1+\rho)}$.*

PROBLEM L.†

§ 331. To determine the present value of a given sum payable on the decease of A and B, provided another life C dies before B.

SOLUTION.

The payment of the given sum at the end of the first year will depend on the happening of one event only; viz. the extinction of all the lives in that year, C having died before B: the probability of this event is $\frac{(a-a')(b-b')(c-c')}{2abc}$; which being multiplied by $s(1+\rho)^{-1}$, will give the expectation of receiving the sum at the end of the first year.

But in the second and following years, the given sum

* Mr. Morgan makes the value of the sum in this case equal to $\frac{S(r-1)}{r} \times (V-CC-C-CCC)$; that is, in the present notation, equal to $s \times \frac{1-\rho(A+AA+AAA)}{(1+\rho)}$: an error which has also

been faithfully transcribed into two successive editions of Dr. Price's treatise. See *Phil. Trans.* for 1794, part II, page 245: and Dr. Price's *Obs. on Rev. Pay.* vol. 1, page 395.

† *Phil. Trans.* for 1794, Prob. 4, page 245; and for 1800, Prob. 7, page 40: which two *might* have been comprised in one problem.

may be received on the happening of either of five different events: 1° that all the lives fail in the year, C having died before B: 2° that A and B fail in the year, and that C fails in either of the preceding years: 3° that B and C both fail in the year, C having died first, and that A dies in either of the preceding years: 4° that only B dies in the year, and that A and C both die in either of the preceding years: 5° that only A fails in the year, and *that B and C both fail in either of the preceding years, B having died last.* The probabilities of the happening of these several events in the second year will be respectively $\frac{(a-a'')(b-b'')(c-c'')}{2abc}$, $\frac{(a-a'')(b-b'')}{ab} \left(1 - \frac{c'}{c}\right)$, $\frac{(b-b'')(c-c'')}{2bc} \left(1 - \frac{a'}{a}\right)$, $\frac{b-b''}{b} \times \left(1 - \frac{a'}{a}\right) \cdot \left(1 - \frac{c'}{c}\right)$, and $\frac{a-a''}{2a} \left(1 - \frac{b'}{b}\right) \cdot \left(1 - \frac{c'}{c}\right)$: which, being added together and multiplied by $s(1+e)^{-2}$, will give the expectation of receiving the sum at the end of the second year. In like manner may be found the expectation of receiving the sum at the end of the third and every subsequent year during the continuance of the oldest life: and, if these several expectations be reduced to their lowest terms and arranged under each other, they will form seventeen collateral series; the sum of the first n terms of which will vary according to the seniority of the lives concerned.

§ 332. Case 1. Let A be the oldest of the three lives. In this case, the several collateral series here

mentioned must be continued to the utmost extent of human life; because the life of A is involved in each perpendicular series: whence, its value will be equal

to s multiplied into $\frac{1-\rho A}{2(1+\rho)} + \frac{1-\rho B}{(1+\rho)} - \frac{1+AB}{2(1+\rho)} +$

$AB - \frac{1+AB}{2(1+\rho)} \cdot \frac{a}{a} - \frac{1+AC}{2(1+\rho)} + \frac{1+AC}{2(1+\rho)} \cdot \frac{a}{a} - \frac{1-\rho BC}{2(1+\rho)} +$

$\frac{1+BC}{2(1+\rho)} \cdot \frac{b}{b} - BC \cdot \frac{b}{2b} + \frac{1-\rho ABC}{2(1+\rho)} + ABC \cdot \frac{b}{2b} -$

$ABC \cdot \frac{c}{2c}$. But this expression may be reduced to

$\frac{1-\rho(A+2B-2AB-BC+ABC)+AB-AC}{2(1+\rho)} - \frac{(AB-AC)a}{2(1+\rho)a}$

$+ \frac{1}{2b} \times \left[\frac{(1+BC)b}{(1+\rho)} - (BC-ABC)b \right] - ABC \cdot \frac{c}{2c};$

and which I shall denote by $\mathbf{M}$. Consequently, when A is the oldest life, the required value will be represented by $s \times \mathbf{M}$.

§ 333. *Case 2.* Let B be the oldest of the three lives. In this case, the sum of the first n terms of the several collateral series above alluded to will be equal to $\mathbf{M} - \frac{1-\rho A}{2(1+\rho)} \cdot \frac{a}{a} (1+\rho)^{-n} + \frac{1+AC}{2(1+\rho)} \cdot \frac{a\gamma}{ac} \times (1+\rho)^{-n} - \frac{1+AC}{2(1+\rho)} \cdot \frac{a\gamma}{ac} (1+\rho)^{-n}$. But after the decrease of B, the sum of the expectations for all the subsequent years will be more correctly expressed, as in the second case to Prob. XL, by $s(1-\phi) \times \frac{1-\rho A}{(1+\rho)} \cdot \frac{a}{a} (1+\rho)^{-n}$; and which, being added to the

sum of the first n terms of the several collateral series above mentioned, will make the total present value of the given sum, when B is the oldest life, equal to $s[M + x]$.

§ 334. *Case 3.* Let C be the oldest of the three lives. In this case, the first n terms of the several collateral series above alluded to, will be equal to $M - \frac{1-\rho A'}{2(1+\rho)} \cdot \frac{\alpha}{a} (1+\rho)^{-n} - \frac{1-\rho B'}{(1+\rho)} \cdot \frac{\beta}{b} (1+\rho)^{-n} + \frac{1+A'B'}{2(1+\rho)} \cdot \frac{\alpha\beta}{ab} (1+\rho)^{-n} - A'B' \cdot \frac{\alpha\beta}{ab} (1+\rho)^{-n} + \frac{1+A'B'}{2(1+\rho)} \times \frac{\alpha\beta}{ab} (1+\rho)^{-n}$. But, after the decease of C , the chance of receiving the given sum at the end of any subsequent year will depend on the happening of three different events: 1° that A and B both fail in the year: 2° that only B fails in the year, A having died in either of the preceding years: 3° that only A fails in the year, B having died after C in either of the preceding years. The probabilities of the happening of these several events in the $(n+1)^{st}$ year are respectively $\frac{(\alpha-\alpha')(\beta-\beta')}{ab}$, $\frac{\beta-\beta'}{b}(1-\frac{\alpha}{a})$ and $\frac{\alpha-\alpha'}{a}(1-\phi-\frac{\beta}{b})$: which, being added together and multiplied by $s(1+\rho)^{-(n+1)}$, will give the expectation of receiving the sum at the end of the $(n+1)^{st}$ year.

In like manner we may find the $(n+2)^{nd}$, and all the subsequent expectations to the utmost extremity of human life: the sum of all which will be equal to

s multiplied into $(1-\phi) \times \frac{1-\phi A'}{2(1+\phi)} \cdot \frac{\alpha}{a} (1+\phi)^{-n} + \frac{1-\phi B'}{(1+\phi)} \cdot \frac{\beta}{b} (1+\phi)^{-n} - \frac{1+A'B'}{(1+\phi)} \cdot \frac{\alpha\beta}{ab} (1+\phi)^{-n} + A'B' \times \frac{\alpha\beta}{ab} (1+\phi)^{-n}$; and which, being added to the first n terms of the several collateral series above found, will make the total present value of the given sum, when C is the oldest life, equal to $s[\mathbf{M} + z]$.

COROLLARY.

§ 335. When all the lives are equal, or of the same age A , the last three quantities in the formula denoted by $\mathbf{M}$ destroy each other, and the general expression in this case will become $s \times \frac{1-\phi(3A-3AA+AAA)}{2(1+\phi)}$; that is, equal to half the present value of the given sum, to be received on the extinction of the longest of the three lives.

SCHOLIUM.

§ 336. Since the present values of the sum, as found by the preceding problem, and by the present problem, are together equal to $s \times \frac{1-\phi(A+B-AB)}{(1+\phi)}$; or to the present value of the given sum to be received on the extinction of the longest of the two lives A, B ; it follows that the present value of one of them being found, the present value of the other may be easily determined by subtracting such value from the general expression here given.

CHAPTER IX.

On M. De Moivre's Hypothesis.

§ 337. IF the decrements of life, in any table of observations, were equal and uniform from birth to the utmost extent of human life, the number of persons living at every succeeding age, as shown by that table, would be the terms of a decreasing arithmetical progression. Now, though this fact does not take place throughout the *whole* duration of life; yet, in several of the tables of observations (particularly in the *Northampton* table) it will be seen that, towards the middle ages, the decrements are, for many years together, constant and uniform; or at least very nearly so: and it was upon this view of the subject that M. De Moivre founded his ingenious hypothesis *that the decrements of life are in arithmetical progression*.* For, he imagined that the deficiencies in one part of his scale would be compensated by the excesses of the other part: and that though the values deduced from this principle might not be strictly true, yet the error could not be very material. In order to accom-

* The same author also assumed as an hypothesis that the decrements of life were in a *geometrical* ratio: a principle which he applied to the valuation of annuities on *joint* lives. But this theory is too erroneous to be insisted on.

modate this hypothesis to the method of determining the value of Life Annuities, it was necessary to suppose the extent of life confined to a certain period of time, which he (for reasons which subsequent observations have shown not to be sufficiently well founded) has taken at 86: remarking, however, at the same time, that the instances of persons living beyond that period are so few as not to be worth notice in such a general view of the subject.

Much obloquy has of late years been thrown on this hypothesis by Dr. Price and Mr. Morgan; the latter of whom has been particularly severe in his comments on its use and application. It is true that more recent discoveries have shown that it cannot *always* be depended upon: and the great share which these gentlemen have had in deducing the values of annuities from *real observations*, and thereby superseding the use of the hypothesis, may be some extenuation for the high tone they have assumed on this occasion. Nevertheless, the hypothesis itself is *still* of great and extensive use in the doctrine of annuities; and will ever remain a monument of the ingenuity and abilities of its illustrious inventor.

§ 338. The reader must be readily aware that the great advantage of M. De Moivre's hypothesis is the remarkable facility which it affords for determining the present value of annuities on any single or joint lives. The method of deducing such values from the first problem of the present work (see page 29) is

extremely laborious; since the numerators of the fractions, which form the series there given, do not decrease in any regular manner. But, when such numerators are in arithmetical progression; the sum of the whole series may be expressed in a general formula; and its value may then be found almost as soon as one term only of the other series just alluded to.

Although M. De Moivre, in the application of his principles to practical purposes, has assumed 86 as the extreme period of human life, (that is; that, out of 86 persons born, *one* of them will die annually till they are all extinct) yet this number does not necessarily follow from his hypothesis: for any other number, which might be found more agreeable to real observations, may be substituted in lieu thereof. Assuming, however, the same period as De Moivre, it is evident that the number of persons living at any age will be equal to the number of years between that age and 86: this number he calls the *complement* of life. Consequently *a*, *b*, *c*, &c will, upon this hypothesis, be equal to the complement of the lives *A*, *B*, *C*, &c, respectively: or, equal to the difference between their respective ages and 86.

§ 339. This being premised, it will be readily seen that the expression $\frac{1}{a} \left[\frac{a^I}{(1+\rho)} + \frac{a^{II}}{(1+\rho)^2} + \frac{a^{III}}{(1+\rho)^3} + \&c \right]$ which, by Prob. I, cor. 1, denotes the value of an annuity on the life *A*, will become

$\frac{1}{a} \left[\frac{a-1}{(1+\rho)} + \frac{a-2}{(1+\rho)^2} + \frac{a-3}{(1+\rho)^3} + \dots + \frac{a-a}{(1+\rho)^a} \right]$.
 Now, this series may be divided into two others, the first of which is $\left[\frac{1}{(1+\rho)} + \frac{1}{(1+\rho)^2} + \frac{1}{(1+\rho)^3} + \dots + \frac{1}{(1+\rho)^a} \right]$; and the second, which is to be subtracted, is $\frac{1}{a} \left[\frac{1}{(1+\rho)} + \frac{2}{(1+\rho)^2} + \frac{3}{(1+\rho)^3} + \dots + \frac{a}{(1+\rho)^a} \right]$. But the former of these, which I shall denote by y , is equal to the present value of an annuity certain for the term a ; and the latter is equal to $\frac{1}{a} \times \frac{1+y-(a+1) \cdot (1+\rho)^{-a}}{\rho}$;* consequently the total present value of the annuity is, upon this hypothesis, equal to $y - \frac{1+y-(a+1) \cdot (1+\rho)^{-a}}{a\rho} = \frac{a-(1+\rho)y}{a\rho}$.†

§ 340. In like manner, the expression $\frac{1}{ab} \left[\frac{a'b}{(1+\rho)} + \frac{a''b}{(1+\rho)^2} + \frac{a'''b}{(1+\rho)^3} + \dots \right]$, which denotes the value of an annuity on the two joint lives A B, will become, upon this hypothesis, equal to $\frac{1}{ab} \left[\frac{(a-1) \cdot (b-1)}{(1+\rho)} + \frac{(a-2) \cdot (b-2)}{(1+\rho)^2} + \frac{(a-3) \cdot (b-3)}{(1+\rho)^3} + \dots + \frac{(a-b) \cdot (b-b)}{(1+\rho)^b} \right]$;

* See my *Doctrine of Interest and Annuities*, page 89: where the method of summing this and the following series is pointed out.

† See Dodson, vol. ii. Ques. 56. Simpson, Prob. 1, cor. 5. De Moivre, page 310.

a series which is to be continued for b terms only, B being the oldest of the two lives. But, if the numerators of the fractions composing this series be expanded by multiplication, the whole expression may be divided into three other distinct series: viz.

$$\begin{aligned} & \left[\frac{1}{(1+\rho)} + \frac{1}{(1+\rho)^2} + \frac{1}{(1+\rho)^3} + \dots + \frac{1}{(1+\rho)^b} \right] - \\ & \frac{a+b}{ab} \left[\frac{1}{(1+\rho)} + \frac{2}{(1+\rho)^2} + \frac{3}{(1+\rho)^3} + \dots + \frac{b}{(1+\rho)^b} \right] \\ & + \frac{1}{ab} \left[\frac{1}{(1+\rho)} + \frac{4}{(1+\rho)^2} + \frac{9}{(1+\rho)^3} + \dots + \frac{b^2}{(1+\rho)^b} \right]. \end{aligned}$$

The first of these is equal to the present value of an annuity for the term b , and I shall denote it by y : the second is (from what has been said above) equal to $-\frac{a+b}{ab} \times \frac{1+y-(b+1) \cdot (1+\rho)^{-b}}{\rho}$: and the third is equal to $\frac{1}{ab\rho} \left[1+y+2 \times \frac{1+y-(b+1) \cdot (1+\rho)^{-b}}{\rho} - (b+1)^2 \times (1+\rho)^{-b} \right]$: consequently the total present value of the annuity for two joint lives is equal to $\frac{1}{\rho} - \frac{1+\rho}{a\rho} \times \left[\frac{y}{b} \left(a-b-1-\frac{2}{\rho} \right) + \frac{2}{\rho} \right]$.*

§ 341. By a similar method of reasoning it will be found that the expression $\frac{1}{a'c} \left[\frac{a'b'c'}{(1+\rho)} + \frac{a''b''c''}{(1+\rho)^2} + \frac{a'''b'''c'''}{(1+\rho)^3} + \&c \right]$, which denotes the value of an an-

* See Dodson, vol. ii. Ques. 64. Simpson, Prob. 1, cor. 5. Price, note (L). I shall here take the opportunity of observing that the strictures of Dr. Price on M. De Moivre's rules for calcu-

nuity on the three joint lives A B C, will become, upon this hypothesis, equal to $\frac{1}{abc} \left[\frac{(a-1).(b-1).(c-1)}{(1+\rho)} + \frac{(a-2).(b-2).(c-2)}{(1+\rho)^2} + \frac{(a-3).(b-3).(c-3)}{(1+\rho)^3} + \dots \dots \dots \frac{(a-c).(b-c).(c-c)}{(1+\rho)^c} \right]$; a series which is to be continued for c terms only, C being now the oldest life. But the whole of this series may, by expansion, be converted into the four following series: viz. $\left[\frac{1}{(1+\rho)} + \frac{1}{(1+\rho)^2} + \frac{1}{(1+\rho)^3} + \dots \dots \dots \frac{1}{(1+\rho)^c} \right] - \frac{ab+ac+bc}{abc} \times \left[\frac{1}{(1+\rho)} + \frac{2}{(1+\rho)^2} + \frac{3}{(1+\rho)^3} + \dots \dots \dots \frac{c}{(1+\rho)^c} \right] + \frac{a+b+c}{abc} \left[\frac{1}{(1+\rho)} + \frac{4}{(1+\rho)^2} + \frac{9}{(1+\rho)^3} + \dots \dots \dots \frac{c^2}{(1+\rho)^c} \right] - \frac{1}{abc} \left[\frac{1}{(1+\rho)} + \frac{8}{(1+\rho)^2} + \frac{27}{(1+\rho)^3} + \dots \dots \dots \frac{c^3}{(1+\rho)^c} \right]$. Now, the first of these series is equal to the present value of an annuity certain for the term c ; and I shall denote it by y ; and the remaining ones may be summed in the same manner

lating the value of annuities on joint lives (inserted in the 4th Chapter of his *Obs. on Rev. Pay.*) allude altogether to an *approximation* given by De Moivre as deduced from his hypothesis of a *geometrical* ratio, and not to the *correct* expression. The formula in the text will oftentimes give the same values as those deduced from the *Northampton* Table; as may be seen by a comparison with the Tables at the end of the present work: and I shall show in the sequel that these formulæ may be used with good effect in determining, with a tolerable degree of correctness, the value of annuities deduced from the *real probabilities* of life. See § 346.

as in the preceding cases: whence the total present value of an annuity on the three joint lives is equal to $\frac{1}{\rho} - \frac{1+\rho}{a\rho} \left[\frac{1}{\rho} (2b + 2a - c - 3 - \frac{6}{\rho}) - \frac{2y}{c\rho} (b + a - 2c - 3 - \frac{3}{\rho}) + \frac{y}{c} (b - c - 1) \cdot (a - c - 1) \right]$.*

§ 342. But, since the publication of so many accurate tables of the values of annuities, deduced from *real observations*, these formulæ have become of little or no use; and are seldom resorted to, unless it is required to find a near value of an annuity on any given lives whose ages are not inserted in those tables. Nevertheless the hypothesis itself is still of great use in the doctrine of annuities; and facilitates very much several calculations arising from this subject: particularly in those cases where the contingency continues for a given *term* only; as will evidently appear from the following problem.

PROBLEM LI.†

§ 343. To find the value of an annuity on a given life, during part of whose existence the decrements of life may be considered as equal.

* See Dodson, vol. ii. Ques. 69. Simpson, Prob. 1, cor. 5.

† De Moivre, page 341. Simpson, Prob. 1, cor. 8. Price, note (N). Dodson, vol. ii. Ques. 68. This last author has also given a method of solution (Question 61) provided the decrements of life be divided into *several* arithmetical progressions, each differing from one another.

SOLUTION.

Let A be the given life, and n the interval of equal decrements: also let a be the number of persons living at the age of A , and α the number of persons living at an age n years older; and let the uniform decrements of life be denoted by δ . Then will the value of an annuity on this life be denoted by the

series $\frac{1}{a} \left[\frac{a-\delta}{(1+\rho)} + \frac{a-2\delta}{(1+\rho)^2} + \frac{a-3\delta}{(1+\rho)^3} + \dots \dots \frac{a-n\delta}{(1+\rho)^n} \right.$
 $\left. + \frac{\alpha'}{(1+\rho)^{n+1}} + \frac{\alpha''}{(1+\rho)^{n+2}} + \frac{\alpha'''}{(1+\rho)^{n+3}} + \&c \right]$; which

may be divided into two parts: viz. the series

$\frac{1}{a} \left[\frac{a-\delta}{(1+\rho)} + \frac{a-2\delta}{(1+\rho)^2} + \frac{a-3\delta}{(1+\rho)^3} + \dots \dots \frac{a-n\delta}{(1+\rho)^n} \right]$
 $+ \frac{1}{a} \left[\frac{\alpha'}{(1+\rho)^{n+1}} + \frac{\alpha''}{(1+\rho)^{n+2}} + \frac{\alpha'''}{(1+\rho)^{n+3}} + \&c \right]$. The

latter of these is, by Prob. I, cor. 3, equal to $(A)^d$, or equal to the value of an annuity on the life A *deferred* for n years: and the former may be divided into two

others; viz. $\left[\frac{1}{(1+\rho)} + \frac{1}{(1+\rho)^2} + \frac{1}{(1+\rho)^3} + \dots \dots \right.$

$\left. \frac{1}{(1+\rho)^n} \right] - \frac{\delta}{a} \left[\frac{1}{(1+\rho)} + \frac{2}{(1+\rho)^2} + \frac{3}{(1+\rho)^3} + \dots \dots \right.$

$\left. \frac{n}{(1+\rho)^n} \right]$: the sum of which may be readily found,

from what has preceded, to be equal to $y -$

$\frac{\delta}{a\rho} \left[(1+\rho)y - n(1+\rho)^{-n} \right]$. Consequently the total

value of the annuity will be $y - \frac{\delta}{a\rho} \left[(1+\rho)y -$

$n(1+\rho)^{-n}] + (A)^d$. But, since δ is always equal to $\frac{a-\alpha}{n}$;* and since $(A)^d$ is equal to $A \times \frac{\alpha}{a}(1+\rho)^{-n}$; this expression may be rendered more convenient for practice by making those substitutions: whence the value will become $\frac{1}{\rho} - \frac{(a-\alpha) \cdot (1+\rho)y}{an\rho} - \frac{\alpha(1+\rho)^{-n}}{a\rho} + A \times \frac{\alpha}{a}(1+\rho)^{-n} = \frac{1}{\rho} - \frac{1}{a} \left[\frac{a-\alpha}{n\rho}(1+\rho)y - \alpha \left(\frac{1}{\rho} - A \right) \cdot (1+\rho)^{-n} \right]$. Now it appears by the *Northampton* table of observations that the decrements of life are *nearly* uniform during the whole interval from the age of 20 to the age of 80: if therefore the value of an annuity on a life aged 80 be once calculated, the value of an annuity on a life of any intermediate age may be easily deduced from the formula here given; and will agree very nearly with the true values.

Example. Let the rate of interest be 5 per cent; in which case the value of an annuity on a life of 80 years of age will be 3.515: and, if from this we wish to deduce the value of an annuity on a life of 20 years of age, the formula will become $20 - \frac{1}{5132} \left[\frac{5132-469}{60 \times .05} \times 1.05 \times 18.9293 - 469 \times (20 - 3.515) \times .0535 \right] = 14.061$: and which is very near 14.007, or the true value as shown by the tables.

* This may generally be assumed as the true value, without material error, even when the decrements are not exactly regular.

Now, when it is considered that the value, here found, could not have been deduced from the corollary in page 29, without the actual calculation of *sixty* terms of the series there given, the utility of this formula will be manifest.

COROLLARY.

§ 344. If the value of the annuity is required for a given *term* ($=n$); and that term happens to be wholly within the interval of equal decrements as shown by any given table of observations; the series $\frac{1}{a} \left[\frac{a-\delta}{(1+\rho)} + \frac{a-2\delta}{(1+\rho)^2} + \frac{a-3\delta}{(1+\rho)^3} + \dots \dots \dots \frac{a-n\delta}{(1+\rho)^n} \right]$ $= y - \frac{a-\alpha}{an\rho} \left[(1+\rho)y - n(1+\rho)^{-n} \right]$ will denote the *exact* value in such case; and is a formula of considerable utility when we are not possessed of any tables of the value of annuities deduced from such observations. Or, if the decrements are very nearly equal, the formula will not differ materially from the true expression.

Example. Suppose it were required to find the value of an annuity for 20 years on a life aged 20, reckoning interest at .5 per cent and the probabilities of living as at *Northampton*. In this case the formula would become $12.4622 - \frac{5132-3635}{5132 \times 20 \times .05} \times [1.05 \times 12.4622 - 20 \times .3769] = 10.844$: which is the *correct* value of a temporary annuity for 20 years on a life aged 20; because it will be seen, by Table XXV,

that from the age of 20 to the age of 40 the decrements of life are equal. The value, deduced from the rule in page 46, is 10·847.

§. 345. This problem and its corollary will serve to show the useful purposes to which M. De Moivre's hypothesis may be occasionally appropriated, and the method of applying it whenever an opportunity occurs. But since the decrements of life are most irregular in the younger and in the latter periods of existence, and are uniform (or nearly so) during the middle ages only; it will be found that this hypothesis cannot in all cases be safely used, unless in deducing the value of annuities or assurances for *terms*. In this respect it is of singular utility, and will be often found to save a laborious calculation, as I have already pointed out in the notes to some of the preceding problems: and, therefore, it will be unnecessary to enlarge more upon the subject in this place. It is in this manner that Mr. Morgan has condescended to use it: but it is done clandestinely, and (I know not for what reason) by a previous *denial* of the fact.*

I would here observe that the rule for determining the value of such annuities as depend on the whole continuance of any number of lives out of any other number of lives, or such as are in reversion or depending upon survivorships, and in general for deter-

* See Price's *Obs. on Rev. Pay.* vol. 1, page 61, note e: also the note in page 192 of the present work.

mining such problems as are contained in the second, third and fourth Chapters of this work are the same on M. De Moivre's hypothesis, as where they are deduced from real observations. For, in each case, the solutions to such questions are obtained from *Tables* which show the values of annuities on single and joint lives: and therefore the merit or demerit of M. De Moivre's hypothesis, as far as regards the value of *annuities*, will rest on the fundamental propositions above given.

But in deducing the value of *assurances*, or reversionary sums, his rules are certainly much more simple than when deduced from real observations: and it would be a fortunate circumstance if his hypothesis could be depended upon throughout the *whole duration* of life. As this, however, is not the case, we must be content with the facility which it oftentimes affords us of determining, in many cases, a very near value of such assurances for *terms*: as I have already fully explained in the sixth and eighth Chapters.*

On the Method of approximating to the Value of Life Annuities.

§ 346. I cannot dismiss this chapter, however, without noticing the utility and convenience of the *formulæ* arising from M. De Moivre's hypothesis, in enabling us also to deduce (from the values of an-

* See the notes in pages 150, 192, 205 and 213.

nuities on single or joint lives at any one rate of interest) the values of annuities on the same lives, at any *other* rate of interest. In order to explain this method, I would observe that, according to M. De Moivre's hypothesis, the *expectation* of any life is equal to half the complement of such life:* consequently, the complement of any life is equal to twice its expectation. If, therefore, we substitute *twice the expectation* of any life deduced from real observations, instead of the quantities a , b , or c , in the general formulæ in pages 316, 317 and 319, the values thence arising will in most cases be much nearer the true values of annuities deduced from such observations, than when a , b , or c , is taken equal to the complement of such life according to M. De Moivre's hypothesis.† Or, (which is all that is required in the present instance) the difference between the values of an annuity on any single or joint lives, deduced in this manner from the *expectations* of life, at any two rates of interest, will be nearly the same as the difference between the *correct* values of a similar annuity, at the same rates of interest, deduced from *real observations*. Consequently, when the value of an annuity according to any one rate of interest is given, we may readily obtain a very near value of a similar annuity at any other rate, by means of the first difference here alluded to: as will be evident from the following general rule.

* See the note in page 50.

† See the observations at the end of the following note.

Call the *correct* value, computed at any one rate of interest, the *FIRST* value.

Call the value deduced from the *expectations* of life, at the *same* rate of interest, the *SECOND* value.

Call the value deduced from the *expectations* of life, at any *other* rate of interest, the *THIRD* value.

Then, the *difference* between the *SECOND* and *THIRD* values subtracted from, or added to, the *FIRST* value (according as the *SECOND* is greater or less than the *THIRD*) will be the near value of the annuity at the *other* rate of interest required.

Example 1. What is the near value of an annuity on a life aged 20 years at 4 per cent interest, deduced from the correct value at 5 per cent, and according to the *Northampton* observations?

The *first*, or correct value at 5 per cent is, by Table XXVII, equal to 14.007. The *second* value, deduced at the same rate of interest from the expectation of that life by the formula in page 316, is equal to $\frac{66.86 - 1.05 \times 19.234}{66.86 \times .05} = 13.959$.* The *third* value, deduced from the same formula, at 4 per cent, is equal

* When twice the expectation is equal to a whole number with a decimal added (as is commonly the case), the value of an annuity for that term may be best computed in the following manner. Suppose the number of years (as in the present case) to be 66.86. The value of an annuity for 66 years is, by Table LIX, equal to 19.201; and the value of an annuity for 67 years is 19.239. The difference between these two values is .038; which, being multiplied by the decimal .86, and the product .033 added to the least

to $\frac{66.86 - 1.04 \times 23.183}{66.86 \times .04} = 15.984$. Therefore $14.007 + (15.984 - 13.959) = 16.032$ will be the near value required : and which differs only an unit in the last figure from the true value as given in Table XXVII.

§ 347. The same principles will apply to the case of two *joint* lives : and it will be found that in both cases the deduced values are sometimes nearly the same as the correct values ; that generally they do not differ more than a 20th or 30th part of a year's purchase ; that in *joint* lives they differ less than in single lives ; and that they come equally near to each other *whatever* the rates of interest are.*

of the two values, will give 19.234 for the value of the annuity for 66.86 years.

The second and third values here obtained (that is, 13.959 and 15.984) will be found, on a comparison with the values in Table XXVII, to be much nearer the *true* values than those obtained from M. De Moivre's hypothesis, as given in Table XLIX. Consequently this first step of the process will show that M. De Moivre's *formula* (as given in page 316) may sometimes be applied, with good effect, to find, in an expeditious manner by *one* operation, a near value of an annuity deduced from real observations.

* See a variety of examples, in proof of these assertions, in Dr. Price's *Obs. on Rev. Pay.* vol. 1, page 231. The same author remarks that " these deductions, in the case of single lives particularly, are so easy, and give the true values so nearly, that it " will be scarcely ever *necessary* to calculate the exact values (according to any given observations) for more than *one* rate of " interest." But, however convenient the above rules may be in our present dearth of useful Tables, they by no means remove

On the Value of increasing Life Annuities.

§ 348. The hypothesis of M. De Moivre furnishes us likewise with a convenient and useful formula for determining the value of *increasing* Life Annuities: that is, of £1, £2, £3, &c (or any multiples of those sums) payable at the end of 1, 2, 3, &c, years respectively, if the given life A be then in existence. For, the series expressing such value will (from what has been said in page 316) be evidently equal to $\frac{1}{a} \left[\frac{a-1}{(1+\rho)} + \frac{2(a-2)}{(1+\rho)^2} + \frac{3(a-3)}{(1+\rho)^3} + \dots \dots \frac{a(a-a)}{(1+\rho)^a} \right]$, and which may be divided into the two following series,

$$\left[\frac{1}{(1+\rho)} + \frac{2}{(1+\rho)^2} + \frac{3}{(1+\rho)^3} + \dots \dots \frac{a}{(1+\rho)^a} \right] - \frac{1}{a} \left[\frac{1^2}{(1+\rho)} + \frac{2^2}{(1+\rho)^2} + \frac{3^2}{(1+\rho)^3} + \dots \dots \frac{a^2}{(1+\rho)^a} \right].$$

In order to abridge the subsequent process, let us make

$$\left[\frac{1}{(1+\rho)} + \frac{1}{(1+\rho)^2} + \frac{1}{(1+\rho)^3} + \dots \dots \frac{1}{(1+\rho)^a} \right] = y$$

$$\left[\frac{1}{(1+\rho)} + \frac{2}{(1+\rho)^2} + \frac{3}{(1+\rho)^3} + \dots \dots \frac{a}{(1+\rho)^a} \right] = x$$

$$\left[\frac{1^2}{(1+\rho)} + \frac{2^2}{(1+\rho)^2} + \frac{3^2}{(1+\rho)^3} + \dots \dots \frac{a^2}{(1+\rho)^a} \right] = w$$

then will the sum of the two series above given, or the value of the annuity required, be denoted by $x - \frac{w}{a}$.

the necessity of calculating, at *several* rates of interest, any new Tables that may be hereafter formed: and I should hope that no one, who may at any future time undertake this laborious task, will be influenced by so weak and so ill-judged an excuse. The object and real utility of Tables, of any kind, is to save time and labour, and to prevent the occurrence of errors.

But, from what has been said in § 339, it will be seen that the value of an annuity on a single life A is denoted by $y - \frac{x}{a}$; and, from what has been said in § 340, that the value of an annuity on two equal joint lives AA is denoted by $y - \frac{2x}{a} + \frac{w}{aa}$. Therefore $A - AA = y - \frac{x}{a} - y + \frac{2x}{a} - \frac{w}{aa} = \frac{x}{a} - \frac{w}{aa}$; consequently $a(A - AA) = x - \frac{w}{a}$ will be the value of the increasing annuity required: whence we deduce the following rule.

§ 349. *From the value of an annuity on the given life subtract the value of an annuity on two equal joint lives, of the same age with the given life; multiply the difference by the complement (or twice the expectation) of the given life: and the product will be the answer required.*

§ 350. *Example.* Let the given life be 40 years of age; and let the annuity be £1 the first year, £2 the second year, £3 the third year, and so on according to the order of the natural numbers: what is the present value of this annuity, reckoning interest at 4 per cent, and the probabilities of living as observed at *Northampton*?

Here we shall have $A = 13.197$, $AA = 9.820$, and a (or twice the expectation) = 46.16: consequently $46.16 \times (13.197 - 9.820) = 155.882$ will be the value required.

If the annuities had been £10, £20, £30, &c, the present value would be 155.882×10 . Or, if they had been £15, £30, £45, &c, the present value would be 155.882×15 .

But, if the annuity *commences* with a larger sum than £1, and yet increases only by £1 in every year, we must add to the value, above found, the value of an annuity on the given life multiplied into the first payment lessened by unity; and the sum will be the answer. Thus, if the annuities in the first case above mentioned, had been £15, £16, £17, &c, we must multiply 13.197 by 14; and the product, or 184.758, added to 155.882 will give 340.640 for the answer in this case required.

§ 351. These and many other instances (in addition to those already mentioned in various parts of this treatise) might be adduced to show the great utility and convenience of M. De Moivre's hypothesis in a general point of view. The most common cases will convince us that it may always lay claim to a considerable share of merit; but that it is particularly entitled to our approbation in enabling us to conduct our inquiries into many branches of this science where the common analysis is not only exceedingly intricate but sometimes entirely fails; and that it is by no means deserving of the false and ignominious epithets of "wretched" or "absurd."

CHAPTER X.

*On the Value of Annuities payable Half-yearly, &c.
On Half-yearly &c Assurances; and on Annuities
secured by Land.*

§ 352. IN the preceding chapters, the values of annuities have been deduced on the supposition that they are all payable yearly: this is the most usual case. But, as others may occasionally occur, it will be useful to know the limits of the differences which arise in those cases: therefore, that nothing might be wanting on this subject, I shall make no apology for introducing the following investigations.*

If a , a' , a'' , a''' , &c represent the number of persons living at the age of A , and at the age of 1, 2, 3, &c years older than A , agreeably to what has been said

* A person who receives a life annuity *half-yearly* has a double advantage over one who receives the same annuity yearly: for, besides the interest of each half-yearly payment for six months, he has a chance of receiving one half year's payment *more* than if he were paid yearly. In like manner, a person who receives a life annuity *quarterly* has a double advantage over one who receives the same annuity half-yearly: &c, &c. See this subject detailed at full length in Baron Maseres's *Doctrine of Life Annuities*, pages 233—260.

in § 23, then will $\frac{a+a'}{2}$, $\frac{a'+a''}{2}$, $\frac{a''+a'''}{2}$, &c denote the number of persons living at the end of $\frac{1}{2}$, $1\frac{1}{2}$, $2\frac{1}{2}$, &c years from the age of A: which, though perhaps not in *all* cases strictly true, will serve our present purpose, and be as near the real value as we could hope for. Consequently, the present value of an annuity on the

life A, payable *Half-yearly*, is equal to $\frac{1}{2a} \left[\frac{a+a'}{2(1+\rho)^{\frac{1}{2}}} + \frac{a'}{(1+\rho)} + \frac{a'+a''}{2(1+\rho)^{\frac{3}{2}}} + \frac{a''}{(1+\rho)^2} + \frac{a''+a'''}{2(1+\rho)^{\frac{5}{2}}} + \frac{a'''}{(1+\rho)^3} + \&c \right]$:*

a series which may evidently be divided into the two following ones; viz. $\frac{1}{2a} \left[\frac{a+a'}{2(1+\rho)^{\frac{1}{2}}} + \frac{a'+a''}{2(1+\rho)^{\frac{3}{2}}} + \frac{a''+a'''}{2(1+\rho)^{\frac{5}{2}}} + \&c \right] + \frac{1}{2a} \left[\frac{a'}{(1+\rho)} + \frac{a''}{(1+\rho)^2} + \frac{a'''}{(1+\rho)^3} + \&c \right]$. The latter of these is equal to $\frac{A}{2}$; and the

former, which may be divided again into the two following series $\frac{1}{4a(1+\rho)^{\frac{1}{2}}} \left[a + \frac{a'}{(1+\rho)} + \frac{a''}{(1+\rho)^2} + \frac{a'''}{(1+\rho)^3} + \&c \right] + \frac{(1+\rho)^{\frac{1}{2}}}{4a} \left[\frac{a'}{(1+\rho)} + \frac{a''}{(1+\rho)^2} + \frac{a'''}{(1+\rho)^3} + \&c \right]$, is equal to $\frac{1+A}{4(1+\rho)^{\frac{1}{2}}} + \frac{(1+\rho)^{\frac{1}{2}}A}{4}$. Whence

* The reader is supposed to be acquainted with the method of deducing the present value of an annuity *certain* payable half-yearly &c, as explained in my *Doctrine of Interest and Annuities*, Chapter X.

the total present value of the annuity is equal to $\frac{A}{2} + \frac{1+A}{4(1+\epsilon)^{\frac{1}{2}}} + \frac{(1+\epsilon)^{\frac{1}{2}}A}{4} = \frac{2(1+\epsilon)^{\frac{1}{2}}A + 1+A + (1+\epsilon)A}{4(1+\epsilon)^{\frac{1}{2}}} = \frac{2(1+\epsilon)^{\frac{1}{2}} + 2+\epsilon}{4(1+\epsilon)^{\frac{1}{2}}} \times A + \frac{1}{4(1+\epsilon)^{\frac{1}{2}}}.$

But, since the quantity $\frac{2(1+\epsilon)^{\frac{1}{2}} + 2+\epsilon}{4(1+\epsilon)^{\frac{1}{2}}}$ seldom much exceeds unity,* this expression may be taken (without material error) equal to $A + \frac{1}{4(1+\epsilon)^{\frac{1}{2}}}$; and, since $\frac{1}{4(1+\epsilon)^{\frac{1}{2}}}$ is seldom much below $\frac{1}{4}$,† the expression may be still be further reduced to $A + \frac{1}{4}$. That is, if to the value of the annuity payable yearly we add a *quarter of a year's purchase*, the sum will be *very near* the value of the same annuity payable *Half-yearly*. The *exact* values however may be easily determined from the general expression above given.

§ 353. If we wish to determine the present value of a similar annuity payable *Quarterly*, we must take

* When the rate of interest is 2 per cent per annum, the quantity, here alluded to, is equal to 1·000025; and when the rate of interest is 10 per cent per annum, it is equal to 1·000569: whence a judgment may be formed of its value at any intermediate rate.

† When the rate of interest is 2 per cent per annum, this quantity is equal to ·2475; and, when the rate of interest is 10 per cent per annum, it is equal to ·2384: but, it will be seen that as this quantity *decreases*, the one mentioned in the last note *increases*: whence $A + \frac{1}{4}$ will seldom much exceed the true value.

$\frac{3a+a'}{4}$, $\frac{a+3a'}{4}$, $\frac{3a'+a''}{4}$, $\frac{a'+3a''}{4}$, $\frac{3a''+a'''}{4}$, $\frac{a''+3a'''}{4}$, &c to denote the number of persons living at the end of $\frac{1}{4}$, $\frac{2}{4}$, $1\frac{1}{4}$, $1\frac{2}{4}$, $2\frac{1}{4}$, $2\frac{2}{4}$, &c years:* whence, the present value of an annuity on the life A payable *Quarterly* will be equal to $\frac{1}{4a} \left[\frac{3a+a'}{4(1+\rho)^{\frac{1}{4}}} + \frac{a+a'}{2(1+\rho)^{\frac{1}{2}}} + \frac{a+3a'}{4(1+\rho)^{\frac{3}{4}}} + \frac{a'}{(1+\rho)} + \frac{3a'+a''}{4(1+\rho)^{\frac{1}{2}}} + \frac{a'+a''}{2(1+\rho)^{\frac{3}{4}}} + \frac{a'+3a''}{4(1+\rho)^{\frac{5}{4}}} + \frac{a''}{(1+\rho)^{\frac{3}{2}}} + \&c \right]$.

But, this series may be divided into the four following

$$\begin{aligned} \text{ones: viz. } & \frac{1}{16a(1+\rho)^{\frac{1}{4}}} \left[\frac{3a+a'}{1} + \frac{3a'+a''}{(1+\rho)} + \frac{3a''+a'''}{(1+\rho)^{\frac{3}{4}}} + \&c \right] \\ &= \frac{3(1+A)}{16(1+\rho)^{\frac{1}{4}}} + \frac{A(1+\rho)^{\frac{1}{4}}}{16}; \quad \frac{1}{8a(1+\rho)^{\frac{1}{2}}} \left[\frac{a+a'}{1} + \frac{a'+a''}{(1+\rho)} + \frac{a''+a'''}{(1+\rho)^{\frac{3}{4}}} + \&c \right] \\ &= \frac{1+A}{8(1+\rho)^{\frac{1}{2}}} + \frac{A(1+\rho)^{\frac{1}{2}}}{8}; \quad \frac{1}{16a(1+\rho)^{\frac{3}{4}}} \times \\ & \left[\frac{a+3a'}{1} + \frac{a'+3a''}{(1+\rho)} + \frac{a''+3a'''}{(1+\rho)^{\frac{3}{4}}} + \&c \right] = \frac{1+A}{16(1+\rho)^{\frac{3}{4}}} + \\ & \frac{3A(1+\rho)^{\frac{3}{4}}}{16}; \quad \text{and } \frac{1}{4a} \left[\frac{a'}{(1+\rho)} + \frac{a''}{(1+\rho)^{\frac{3}{4}}} + \frac{a'''}{(1+\rho)^{\frac{5}{4}}} + \&c \right] \\ &= \frac{A}{4}. \quad \text{Whence, the total present value of the} \\ \text{annuity is equal to } & \frac{3(1+A)}{16(1+\rho)^{\frac{1}{4}}} + \frac{A(1+\rho)^{\frac{1}{4}}}{16} + \frac{1+A}{8(1+\rho)^{\frac{1}{2}}} \\ &+ \frac{A(1+\rho)^{\frac{1}{2}}}{8} + \frac{1+A}{16(1+\rho)^{\frac{3}{4}}} + \frac{3A(1+\rho)^{\frac{3}{4}}}{16} + \frac{A}{4} = \end{aligned}$$

* These are the arithmetical means between the number of persons living at the age of A and at the several ages of $\frac{1}{2}$, $1\frac{1}{2}$, $2\frac{1}{2}$, $3\frac{1}{2}$, &c, years older: and are sufficiently near for the purposes here intended.

$$\frac{4(1+\rho)^{\frac{1}{2}} + (4+\rho) \cdot (1+\rho)^{\frac{1}{2}} + 2(2+\rho) \cdot (1+\rho)^{\frac{1}{2}} + 4+3\rho}{16(1+\rho)^{\frac{1}{2}}} \times A + \frac{3(1+\rho)^{\frac{1}{2}} + 2(1+\rho)^{\frac{1}{2}} + 1}{16(1+\rho)^{\frac{1}{2}}}.$$

But, since the fractional quantity, by which A is multiplied, seldom much exceeds unity,* this expression may be taken (without material error) equal to $A + \frac{3(1+\rho)^{\frac{1}{2}} + 2(1+\rho)^{\frac{1}{2}} + 1}{16(1+\rho)^{\frac{1}{2}}}$; and since $\frac{3(1+\rho)^{\frac{1}{2}} + 2(1+\rho)^{\frac{1}{2}} + 1}{16(1+\rho)^{\frac{1}{2}}}$ is seldom much below $\frac{3}{8}$,† the expression may be still further reduced to $A + \frac{3}{8}$. That is, if to the value of the annuity payable yearly we add *three eighths of a year's purchase*, the sum will be *very near* the value of the same annuity payable *Quarterly*. The exact values, however, may be easily determined, as in the former case, from the general expression above given.

§ 354. Upon M. De Moivre's hypothesis the present value of a life annuity payable *Half-yearly* will be denoted by the series $\frac{1}{2a} \left[\frac{a-\frac{1}{2}}{(1+\rho)^{\frac{1}{2}}} + \frac{a-1}{(1+\rho)} + \right.$

* When the rate of interest is 2 per cent per annum, the quantity, here alluded to, is equal to 1·000037; and when the rate of interest is 10 per cent per annum, it is equal to 1·000710; whence a tolerably correct opinion may be formed of its value at any intermediate rate.

† When the rate of interest is 2 per cent per annum, this quantity is equal to ·3719; and when the rate of interest is 10 per cent per annum, it is equal to ·3605: but, as this value decreases the one mentioned in the last note increases; whence $A + \frac{3}{8}$ will seldom much exceed the true value.

$$\frac{a-\frac{1}{2}}{(1+\rho)^{\frac{1}{2}}} + \frac{a-2}{(1+\rho)^1} + \frac{a-\frac{3}{2}}{(1+\rho)^{\frac{3}{2}}} + \frac{a-3}{(1+\rho)^2} + \dots \dots \dots \frac{a-a}{(1+\rho)^a} \Big] : \text{which may be divided into the two follow-}$$

$$\text{ing ones: viz. } \frac{1}{2} \left[\frac{1}{(1+\rho)^{\frac{1}{2}}} + \frac{1}{(1+\rho)^1} + \frac{1}{(1+\rho)^{\frac{3}{2}}} + \frac{1}{(1+\rho)^2} + \dots \dots \dots \frac{a}{(1+\rho)^a} \right] - \frac{1}{4a} \left[\frac{1}{(1+\rho)^{\frac{1}{2}}} + \frac{2}{(1+\rho)^1} + \frac{3}{(1+\rho)^{\frac{3}{2}}} + \frac{4}{(1+\rho)^2} + \dots \dots \dots \frac{2a}{(1+\rho)^a} \right]. \text{ The first}$$

$$\text{of these is equal to } \frac{1}{2} \times \frac{1-(1+\rho)^{-a}}{(1+\rho)^{\frac{1}{2}}-1} = h;^* \text{ and the}$$

$$\text{second is equal to } -\frac{1}{4a} \times \frac{2h(1+\rho)^{\frac{1}{2}}-2a(1+\rho)^{-a}}{(1+\rho)^{\frac{1}{2}}-1} : \dagger$$

$$\text{consequently the total present value of the annuity is equal to } \frac{1}{2a} \times \frac{a-h(1+\rho)^{\frac{1}{2}}}{(1+\rho)^{\frac{1}{2}}-1}.$$

In like manner, the present value of a life annuity payable *Quarterly* will, upon the same hypothesis, be equal to $\frac{1}{4a} \left[\frac{a-\frac{1}{4}}{(1+\rho)^{\frac{1}{4}}} + \frac{a-\frac{1}{2}}{(1+\rho)^{\frac{2}{4}}} + \frac{a-\frac{3}{4}}{(1+\rho)^{\frac{3}{4}}} + \frac{a-1}{(1+\rho)^1} + \dots \dots \dots \frac{a-a}{(1+\rho)^a} \right] :$ which may be divided into the two fol-

$$\text{lowing series } \frac{1}{4} \left[\frac{1}{(1+\rho)^{\frac{1}{4}}} + \frac{1}{(1+\rho)^{\frac{2}{4}}} + \frac{1}{(1+\rho)^{\frac{3}{4}}} + \frac{1}{(1+\rho)^1} + \dots \dots \dots \frac{1}{(1+\rho)^a} \right] - \frac{1}{16a} \left[\frac{1}{(1+\rho)^{\frac{1}{4}}} + \frac{2}{(1+\rho)^{\frac{2}{4}}} + \frac{3}{(1+\rho)^{\frac{3}{4}}} + \frac{4}{(1+\rho)^1} + \dots \dots \dots \frac{4a}{(1+\rho)^a} \right]. \text{ But, the first}$$

* See my *Doctrines of Interest and Annuities*, page 56.

† *Ibid.* page 89.

of these series is equal to $\frac{1}{4} \times \frac{1 - (1+\rho)^{-a}}{(1+\rho)^{\frac{1}{4}} - 1} = q$;* and

the second is equal to $-\frac{1}{16a} \times \frac{4q(1+\rho)^{\frac{1}{4}} - 4a(1+\rho)^{-a}}{(1+\rho)^{\frac{1}{4}} - 1}$;

consequently the total present value of the annuity is in this case equal to $\frac{1}{4a} \times \frac{a - q(1+\rho)^{\frac{1}{4}}}{(1+\rho)^{\frac{1}{4}} - 1}$.

By a similar process we might find the present value of such annuities payable at any other intervals: but it will be sufficient to show the extreme limit of the increase which arises from thus supposing the annuity to be payable at such smaller intervals: this limit takes place when the annuity is considered as being paid *momently*; in which case the expression becomes

$\frac{a-m}{a \times NL.(1+\rho)}$: where m is equal to $\frac{1 - (1+\rho)^{-a}}{NL.(1+\rho)}$, or, to the present value of an annuity *certain* for the term a , payable *momently*.†

§ 355. If the numerical value of these expressions for half-yearly and quarterly annuities, according to the hypothesis of M. DeMoivre, be compared with those deduced from real observations, they will be found to confirm the accuracy of each other; and to justify the rule which I have before given: namely, that the value of annuities payable *yearly* must be increased nearly

* See my *Doctrine of Interest and Annuities*, page 56.

† *Ibid.* page 58. And I would here observe that I take NL to denote the *Neperean* logarithm of the quantity immediately following it.

$\frac{1}{4}$ of a year's purchase in order to show the value of the same annuities payable *half-yearly*; and that they must be increased nearly $\frac{3}{8}$ of a year's purchase in order to show the value of the same annuities payable *quarterly*; and also that they must be increased by $\frac{1}{2}$ a year's purchase in order to show the value of the same annuities payable *momently*.

§ 356. The reader must observe that, in all these cases, I have had regard only to the true rate of *annual* interest, agreeably to the principles which I have laid down in another work* for determining the value of annuities in general. But such annual rate may always be expressed in terms of the *nominal* rate, by making the substitutions there alluded to,† according as the interest is payable *half-yearly*, *quarterly*, &c: whereby we shall find that, on M. De Moivre's hypothesis, the present value of an annuity, on the life whose complement is a , payable *yearly*, *half-yearly*, *quarterly*, and *momently*, and on the supposition that the interest is also payable at the same periods, will be denoted respectively by $\frac{a-(1+r)y}{ar}$, $\frac{a-(1+\frac{r}{2})h}{ar}$, $\frac{a-(1+\frac{r}{4})q}{ar}$, and $\frac{a-m}{ar}$: where y , h , q , and m will now respectively denote the present values of annuities

* See my *Doctrine of Interest and Annuities*, page 55.

† That is, by substituting $(1+\frac{r}{m})^m - 1$ for r : agreeably to what I have said in my *Doctrine of Interest and Annuities*, § 77, page 54.

certain for the term a , payable *yearly*, *half-yearly*, *quarterly* and *momently*, and on the supposition that the *interest* also is payable at the *same* periods.

If the numerical value of *these* expressions be compared with each other, it will be found that the half-yearly annuities will, upon this principle, be about $\frac{2}{10}$ of a year's purchase, and quarterly annuities about $\frac{3}{10}$ of a year's purchase, more than the value of the same annuities payable yearly; and this is the rule given by Dr. Price for such purpose.* But as the periods of the payments of the *annuity* are totally independent of the periods of the payment of *interest*, and ought not to be confounded together (as I have more fully explained in the tenth Chapter of my *Doctrine of Interest and Annuities*), we shall find that the addition to the tabular values of $\frac{1}{4}$ and $\frac{3}{8}$ respectively, as stated in pages 333 and 335, will be the most correct rule for general use: agreeably to what has been

* See his *Observations on Rev. Pay.* vol. 1, page 246; and the investigations which are there annexed. The two following examples, given by him, will show the real difference which arises in these cases.

Value of an Annuity on a Single Life, Interest 4 per cent.				
Age.	Payable Yearly.	Payable Half-yearly.	Payable Quarterly.	Payable Momently.
36	13.829	14.010	14.101	14.191
61	8.753	8.973	9.072	9.199

already advanced by Mr. Simpson in his *Doctrine of Annuities and Reversions*, page 79.

As the two methods of investigation are however now before the public, the computist may adopt that rule which he conceives to be best suited to the circumstances of the case. Dr. Price had endeavoured to overturn Mr. Simpson's rule without stating the grounds of his dissent, and to substitute his own without explaining the nature and cause of their difference.

§ 357. Hitherto I have considered the differences in the value of annuities on *single* lives only: but it will be evident that, on the supposition that money is improved at a given *annual* rate of interest, the differences will be nearly the same on two *joint* lives, whose value is deduced from real observations, and whose probabilities of living to the end of every half-year are respectively denoted by $\frac{ab + \overset{1}{a}\overset{1}{b}}{2ab}$, $\frac{\overset{1}{a}\overset{1}{b} + \overset{2}{a}\overset{2}{b}}{2ab}$, $\frac{\overset{2}{a}\overset{2}{b} + \overset{3}{a}\overset{3}{b}}{2ab}$, &c: which probabilities, though not strictly correct, may answer the present purpose.

On M. De Moivre's hypothesis, however, and on the supposition that money can be improved at interest payable at the *same* periods as the annuity, the value of an annuity on the two joint lives whose complements are *a* and *b*, payable *half-yearly*, is accurately expressed by the following series $\frac{1}{2ab} \left[\frac{(a - \frac{1}{2})(b - \frac{1}{2})}{(1 + \frac{1}{2})} + \right.$

$\frac{(a-1).(b-1)}{(1+\frac{r}{2})^2} + \frac{(a-\frac{3}{2}).(b-\frac{3}{2})}{(1+\frac{r}{2})^3} + \frac{(a-2).(b-2)}{(1+\frac{r}{2})^4} + \dots + \frac{(a-b).(b-b)}{(1+\frac{r}{2})^{2b}}$]: the sum of which may, from what has preceded, be easily found equal to $\frac{1}{r} - \frac{2+r}{2ar} \left[\frac{h}{b} \times \left(a-b-\frac{1}{2} - \frac{2}{r} \right) + \frac{2}{r} \right]$.

In like manner the value of an annuity on those lives, payable *quarterly*, is on the same principles equal to $\frac{1}{4ab} \left[\frac{(a-\frac{1}{4}).(b-\frac{1}{4})}{(1+\frac{r}{4})} + \frac{(a-\frac{1}{2}).(b-\frac{1}{2})}{(1+\frac{r}{4})^2} + \frac{(a-\frac{3}{4}).(b-\frac{3}{4})}{(1+\frac{r}{4})^3} + \frac{(a-1).(b-1)}{(1+\frac{r}{4})^4} + \dots + \frac{(a-b).(b-b)}{(1+\frac{r}{4})^{4b}} \right]$ the sum of which is equal to $\frac{1}{r} - \frac{4+r}{4ar} \left[\frac{q}{b} \left(a-b-\frac{1}{4} - \frac{2}{r} \right) + \frac{2}{r} \right]$.

And, if the annuity and the interest are both supposed to be payable *momently*, its value will come out equal to $\frac{1}{r} - \frac{1}{ar} \left[\frac{m}{b} \left(a-b-\frac{2}{r} \right) + \frac{2}{r} \right]$. In which several formulæ, the quantities *h*, *q* and *m* denote the present value of annuities *certain* for the term *b*, payable *half-yearly*, *quarterly* and *momently*, on the supposition that the *interest* also is payable at the *same* periods respectively.*

* The following examples, given by Mr. Morgan, will show the real difference that arises in these cases.

Value of an annuity on two Joint Lives, Interest 4 per cent.				
Ages.	Payable Yearly.	Payable Half-yearly.	Payable Quarterly.	Payable Momently.
20—36	11.227	11.427	11.565	11.629
36—36	10.394	10.600	10.703	10.808
36—61	7.448	7.673	7.793	7.901
61—61	6.144	6.374	6.517	6.602

On Half-yearly &c Assurances.

§ 358. By a similar method of proceeding to that which has been adopted in the former part of this Chapter we might determine the present value of ASSURANCES for every *half-year*, *quarter-year*, &c of human existence. For, if the number of persons living at the end of $\frac{1}{2}$, $1\frac{1}{2}$, $2\frac{1}{2}$, &c years from the age of A be denoted by the same quantities as in page 332, then will the probabilities of such life becoming extinct in the first, second, third, &c, *half-years*, be respectively represented by $\frac{a-a'}{2a}$, $\frac{a-a'}{2a}$, $\frac{a-a''}{2a}$, $\frac{a-a''}{2a}$, $\frac{a-a'''}{2a}$, &c: consequently the present value of an assurance of the sum s on the life A for every *half-year* of human existence will be truly expressed by

$$\frac{s}{2a} \left[\frac{a-a'}{(1+\rho)^{\frac{1}{2}}} + \frac{a-a'}{(1+\rho)} + \frac{a-a''}{(1+\rho)^{\frac{3}{2}}} + \frac{a-a''}{(1+\rho)^2} + \frac{a-a'''}{(1+\rho)^{\frac{5}{2}}} + \&c \right] = \frac{s}{2} \left[1 + (1+\rho)^{\frac{1}{2}} \right] \times \frac{1-\rho^A}{(1+\rho)}.$$

§ 359. In like manner, if the number of persons living at the end of $\frac{1}{4}$, $\frac{3}{4}$, $1\frac{1}{4}$, &c years from the age of A be denoted by the same quantities as in page 334, then will the probabilities of such life becoming extinct in the first, second, third, &c *quarter-years*, be respectively represented by $\frac{a-a'}{4a}$, $\frac{a-a'}{4a}$, $\frac{a-a'}{4a}$, $\frac{a-a'}{4a}$, $\frac{a-a''}{4a}$, $\frac{a-a''}{4a}$, &c: consequently the present value of an assurance of the sum s on the life A for every *quar-*

ter-year of human existence will be truly expressed by

$$\frac{s}{4a} \left[\frac{a-a'}{(1+\rho)^{\frac{1}{4}}} + \frac{a-a'}{(1+\rho)^{\frac{1}{4}}} + \frac{a-a'}{(1+\rho)^{\frac{1}{4}}} + \frac{a-a'}{(1+\rho)^{\frac{1}{4}}} + \frac{a-a''}{(1+\rho)^{\frac{1}{4}}} + \&c \right] = \frac{s}{4} \left[1 + (1+\rho)^{\frac{1}{4}} + (1+\rho)^{\frac{1}{2}} + (1+\rho)^{\frac{3}{4}} \right] \times \frac{1-\rho A}{(1+\rho)} = \frac{s}{4} \times \frac{\rho}{(1+\rho)^{\frac{1}{4}} - 1} \times \frac{1-\rho A}{(1+\rho)}.$$

§ 360. By continuing these subdivisions we shall find that the present value of the same assurance on the life A for every n^{th} part of a year, will be truly expressed by $\frac{s}{n} \left[1 + (1+\rho)^{\frac{1}{n}} + (1+\rho)^{\frac{2}{n}} + \dots \dots \dots \right. \\ \left. \times (1+\rho)^{\frac{n-1}{n}} \right] \times \frac{1-\rho A}{(1+\rho)} = \frac{s}{n} \times \frac{\rho}{(1+\rho)^{\frac{1}{n}} - 1} \times \frac{1-\rho A}{(1+\rho)}.$

Now, when n is infinite, this formula becomes equal to $\frac{s \cdot \rho}{NL \cdot (1+\rho)} \times \frac{1-\rho A}{(1+\rho)}$ * and which consequently denotes the value of the assurance for every *moment* of human existence; that is, the value of the given sum to be received *immediately* on the extinction of the given life.†

* For, in such case $n \left[(1+\rho)^{\frac{1}{n}} - 1 \right]$ is equal to the Neperian logarithm of $(1+\rho)$. See Euler's *Introduct. in Anal. Inf.* vol. 1, chap. 7, § 119; and also what has been said on this subject in my *Doctrine of Interest and Annuities*, page 46.

† Since the Neperian logarithm of $(1+\rho)$ differs but little from $\frac{2\rho}{2+\rho}$, this last formula may be rendered more convenient for practice by means of the expression $\frac{s\rho(2+\rho)}{2\rho} \times \frac{1-\rho A}{(1+\rho)} = s \left(1 + \frac{\rho}{2} \right) \times \frac{1-\rho A}{(1+\rho)}$: which exceeds the value of a yearly assurance, deduced from the rule in page 146, by the quantity $\frac{s\rho}{2} \times \frac{1-\rho A}{(1+\rho)}$

It may be necessary to remark that these values are all deduced from the true *annual* rate of interest; which may be reduced to the *nominal* rate by making the substitution alluded to in the note in page 338.

On Life Annuities secured by Land.

§ 361. A life annuity *secured by land** differs from that kind of life annuity which has been treated of in the preceding part of this work, inasmuch that if the annuitant dies at any time between the stated periods for the payment of the annuity, his heirs are to receive such a sum as will be proportional to the time elapsed between the last payment and his death; whereas in all the cases hitherto considered, if the annuitant dies on the day preceding the time of payment, or sooner, his heirs cannot claim any part or portion of the annuity.

In this case, supposing the annuity payable yearly, the annuitant (since there is the same chance for his dying in one half of any year as in the other) may be considered as having an expectation of half a year's payment more than he would be otherwise entitled to. But the value of the half of £1 to be received on the extinction of any life, A, is by Prob. XXII equal to $\frac{1-\rho A}{2(1+\rho)}$; and, which is the addition that ought to be made to the value of an annuity payable yearly, in

* See on this subject, Dodson, vol. iii. Ques. 1 to 4, and 8 to 14. Price, vol. 1, page 244.

order to obtain its value when secured by land: consequently the value of such annuity is $A + \frac{1-\rho A}{2(1+\rho)}$
 $= \frac{1+(2+\rho)A}{2(1+\rho)}$.

§ 362. In like manner, supposing the annuity payable half-yearly, the annuitant may be considered as having an expectation of a quarter of a year's payment more than he would be otherwise entitled to. But the value of the quarter of £1 to be received on the extinction of the life A in any half year, is, by the formula in page 342, equal to $\frac{[1+(1+\rho)^{\frac{1}{2}}] \times [1-\rho A]}{8(1+\rho)}$; and which is the addition that ought to be made to the value of an annuity payable half-yearly, in order to obtain its value when secured by land. And so on for the additions that ought to be made to the value of an annuity payable quarterly &c. But the *difference* between the value of an annuity payable yearly, *not* secured by land, and the value of an annuity, payable at the same, or at any other intervals, which *is* secured by land, can in no case exceed .05; or half unity.

§ 363. M. De Moivre in his *Doctrine of Chances*, page 338, has given a theorem for finding the value of an annuity *secured by land* and payable yearly, which he deduced by a differential process; a method easily applicable to his hypothesis: and Mr. Dodson in the third volume of his *Mathematical Repository*, page 4, has given another theorem for that purpose (obtained without the aid of that calculus) which

brings out nearly the same answers.* But Mr. Simpson in his *Select Exercises*, page 323, and in the *Supplement* to his *Doctrine of Annuities*, page 70, has given a theorem which shows the value, not of an annuity payable yearly and secured by land, but of an annuity payable *momently* at a given *annual* rate of interest.† The values in all these cases being obtained from M. De-Moivre's hypothesis.

* M. De Moivre's formula is $\frac{1}{\rho} - \frac{y}{a \times NL.(1+\rho)}$; where y denotes the value of an annuity *certain* for the term a , payable yearly. Now the Neperian logarithm of $(1+\rho)$ is very nearly equal to $\frac{2\rho}{2+\rho}$; if, therefore, we substitute this latter quantity, instead of $NL.(1+\rho)$, the above formula will become $\frac{a-y(1+\frac{\rho}{2})}{a\rho}$; which is the same as that given by Mr. Dodson, and which exceeds the value of an annuity not secured by land (as deduced in page 316) by the quantity $\frac{y}{2a}$.

† Mr. Simpson's formula is $\frac{1}{NL.(1+\rho)} - \frac{m}{a \times NL.(1+\rho)} = \frac{a-m}{a \times NL.(1+\rho)}$; which is the same as that given in § 354, for determining the value of a life annuity payable *momently*, at a given *annual* rate of interest; but this is certainly not a correct mode of proceeding in order to find the value of an annuity *secured by land*.

Dr. Price is wrong in asserting that "Mr. Simpson makes the "excess of the value of such an annuity, above the value of an "annuity payable yearly but not secured by land, *double* to the "same excess derived from Mr. Dodson's and M. De Moivre's "rules." The truth is, that not only Dr. Price, but Mr. Simpson himself, appear to have been deceived by the similarity of the symbols employed in the two formulæ compared; without sufficiently considering that those symbols denote *different* quantities.

I would here observe that the formulæ, which I have given above, are the first that have been deduced from real observations; and are much more simple than those deduced from M. De Moivre's hypothesis. But though they readily follow, after the investigations that have been previously entered into; and might easily have been adapted, by preceding writers, to the value of annuities as deduced from such observations; yet those, who have been the most forward to attack the whole of M. De Moivre's principles, have not only suffered his formulæ on this, *and on other subjects*, to remain uncorrected and unproved; but have inserted them in their works as affording a *proper and correct* solution to such cases!!!

CHAPTER XI.

On the Value of Deferred Annuities, Reversionary Annuities, and Assurances, in Annual Payments.

§ 364. IN all those cases of deferred annuities mentioned in Prob. I, cor. 3, and in the corollaries to the subsequent problems, as well as in all cases of Assurances, I have deduced the values of the same in one single payment: but it is oftentimes required to determine such values in *annual* payments. The method of doing which I shall now proceed to show.

In the case of *Deferred* annuities depending on any number of joint lives ABC , the value in one single payment is (by Prob. I, cor. 3) denoted by $(ABC)^d$. Now, if the purchaser of this annuity is desirous of paying for the same by equal *annual* payments during the given term,* those equal payments ought to be such that their total present value shall be equal to the single payment above mentioned: or, in other words, he should pay instead of such sum an *equivalent annuity* during the given term.

§ 365. Let the required annual payment be denoted by p ; and let the value of a temporary annuity

* Such annual payments, however, subject to failure if the given lives become extinct before the end of that period.

on the given lives (that is, of an annuity to continue till the period when the deferred annuity commences) be denoted by $(ABC)^t$: then, since the value of the deferred annuity, or $(ABC)^d$, is to be paid for by equal annual payments during the time such annuity is deferred (subject to failure if any of the given lives become extinct in that period), it is evident that the sum or value of such payments must be equal to the value of an annuity, on the given lives *for such time*, of the yearly value of p : that is, $p(ABC)^t = (ABC)^d$.

This, however, is on the supposition that the first annual payment is not made till the *end* of the first year, and continued at the *end* of every subsequent year till the expiration of the term. But this rarely, if ever, happens: and the usual, if not the invariable method is to advance the first payment *immediately*; and the remaining ones at the *beginning* of each of the following years: so that the number of payments shall be equal to the number of years during which the annuity is deferred. Therefore (since the payment which was supposed, in the preceding case, to be made at the *end* of the *term* is now made at the *beginning*) we must add unity to the value of a temporary annuity for *one year less than** the given term: and this

* Dr. Price, in *all* the cases of annual payments which he has given, says that we must add unity to the value of a temporary annuity *for the given term*: by which means he makes the number of payments to be one more than ever occurs. The reader should particularly observe this in comparing *his* rules with the formula here given.

quantity multiplied by the annual payment will be equal to the value of the deferred annuity. Consequently the formula will, in this case, become $p[1 + (ABC)^{t-1}] = (ABC)^d$; whence, $p = \frac{(ABC)^d}{1 + (ABC)^{t-1}}$ * and whence the following rule.

§ 366. *Divide the value of the Deferred annuity, by unity added to the value of a similar Temporary annuity for one year less than the given term: the quotient will be the annual payment required.*

For examples of the use and application of this rule, see the Scholium to Question VI in Chapter XII.

§ 367. The same rule will apply to the case of deferred annuities depending on the *longest* of two or more lives: see Prob. II, cor. 2. For, if the value of an annuity on the longest of any number of lives be denoted by L , then will the value of a similar *deferred* annuity be denoted by $(L)^d$; and also the value of a similar *temporary* annuity for *one year less than* the given term will be denoted by $(L)^{t-1}$. Consequently, from what has been above said, we shall have $p = \frac{(L)^d}{1 + (L)^{t-1}}$.

* Since $(ABC)^{t-1} = (ABC)^t - \frac{a\beta\gamma}{abc}(1+\rho)^{-n} = ABC - (ABC)^d - \frac{a\beta\gamma}{abc}(1+\rho)^{-n}$; it follows that the formula given in the text may be denoted by $\frac{(ABC)^d}{1 + ABC - [(ABC)^d + \frac{a\beta\gamma}{abc}(1+\rho)^{-n}]}$, which will be oftentimes found very convenient in practice.

For examples of the use and application of this formula, see Question XI in Chapter XII.

In this case, however, it should be particularly observed that if the deferred annuity depends on the *joint* continuance of the given lives to the end of the given term (as mentioned in Prob. II, cor. 3)

the formula will become $p = \frac{(L)^d}{1 + (ABC)^{t-1}}$.

For examples of the use and application of this formula, see the Scholium to Question XI in Chapter XII.

§ 368. A similar method of reasoning will lead us to the true value, in annual payments during the continuance of the given lives, of any *Reversionary* annuity. Thus, let the value of the reversionary annuity, mentioned in the first case in page 71, be denoted by R ; then will the value of the same in annual payments, during the joint continuance of the two lives, be $p = \frac{R}{1 + AP}$. The same formula extends also to the case of *Deferred* reversionary annuities.

For examples of the use and application of this formula, see the Scholium to Question XIII, in Chapter XII: and also Question XVIII, and the Scholium to Question XVIII in that chapter.

But, if the reversionary annuity be *Temporary*, or for a given term only, and such annuity be denoted by $(R)^t$, we shall have $p = \frac{(R)^t}{1 + (AP)^{t-1}}$.

For examples of the use and application of this formula, see Question XIX in Chapter XII.

§ 369. The principles here laid down will likewise extend to all the cases of *Assurances* mentioned in Chapter VI; whether for the whole continuance of life, or for any given term. For, if the present value of an assurance of any given sum be denoted by S , and the present value of a *temporary* assurance of a similar sum be denoted by $(S)^t$; then will the equivalent annuity during the joint continuance of all the lives involved be, in the first case, $p = \frac{S}{1+ABC}$; and in the latter case $p = \frac{(S)^t}{1+(ABC)^{t-1}}$.

It is scarcely necessary to observe, when S denotes the value of an assurance on the *longest* of any number of lives, that ABC will in such case denote the value of an annuity on the *longest* of such lives: agreeably to what has been said in Prob. XXII, cor. 2. And so likewise of any other assurance there alluded to.

For the use and application of the formula, see Questions XXVI, XXVII, and XXIX in Chapter XII.

§ 370. With respect to those assurances which are the subject of Chapter VIII, the annual payment may be divided into three kinds: 1° where such payment is made till the *claim is determined*: 2° where it is made till the *sum becomes due*: 3° where the sum becomes due *at the time* the claim is determined.

Thus in Prob. XXVII, the sum becoming due at the same time that the claim is determined, the value

of the annual payment is obtained by dividing the value of the assurance by unity added to the value of an annuity on the two joint lives AB: that is $p = \frac{S}{1+AB}$.

In Prob. XXVIII, the *claim is determined* on the extinction of the two joint lives: but the sum does not *become due** till the extinction of A's life. Therefore the value of the annual payment till the *claim is determined* will be $p = \frac{S}{1+AB}$: and the value of the annual payment till the sum *becomes due* is $p = \frac{S}{1+A}$.

In Problem XXIX the sum becomes due at the time the claim is determined; and consequently the annual payment is equal to the value of the assurance divided by unity added to the value of an annuity on the three joint lives ABC: that is $p = \frac{S}{1+ABC}$.

In Prob. XXX the claim is not determined, neither does the sum become due, till the extinction of the joint lives AB, and also of the joint lives AC. That is, the annual payment must be made during the continuance of the joint lives AB, and likewise during the continuance of the joint lives AC after the decease of B. The two values will be found equal to $AB+AC-ABC$: consequently we shall have in this case $p = \frac{S}{1+AB+AC-ABC}$.

In Prob. XXXI the annual payment, till the *claim*

* That is, provided the claim is determined *in favour* of the person assuring: and this must be understood in all these cases.

is determined, will be the same as in the last problem: but the value of such payment till the *sum becomes due* is evidently $p = \frac{S}{1+i}$.

In a similar manner we might proceed with respect to the remaining problems in Chapter VIII: but enough has here been said to enable the reader to determine the annual payment in any other case, either of annuities or assurances, that may arise in practice: I therefore shall not detain him with any further remarks on this subject.

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